Defensive Politics and Minority Influence

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Abstract

I analyze a min-max solution to the problem of social choice in a multidimensional policy space called the defensive optimum. The proposed solution applies to a wide variety of models in voting including those concerning poltical advertising, incumbency advantage and interest group politics. The defensive optimum always exists, is unique and is continuous in the density of voter preferences. I analyze other properties by evaluating how its location depends on characteristics of various minority groups in the population. I find that minority groups that are larger, more concentrated and more distant from the mainstream are more influential. Under strong conditions, I find that very small groups have triple the per-member influence of a typical group and that per-member influence declines with size. I also show that distance from the mainstream and group preference concentration are complementary in producing influence, as are distance and size. These facts accord with casual observation: small, concentrated and fringe groups exert disproportionate influence.

1 Introduction

The problem of sincere voting in a multidimensional policy space has been well documented. The well known Median Voter Theorem (Black (1948)) states that if a set of choices can be ordered along a single dimension so that all voters have single-peaked preferences on this ordering, the most preferred policy (MPP) of the median voter can beat any alternative in pair-wise voting. Such a policy is defined to be a Condorcet Winner. A wide variety of models of the political process will result in a Condorcet Winner being chosen, and this makes it an

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appealing concept. Unfortunately, choices cannot generally be ordered along a single dimension so a Condorcet Winner will not typically exist. Plott (1967) showed this for a discrete set of voters as did Davis, DeGroot and Hinich (1972) for a continuous density.[1]

Regardless of whether a Condorcet winner exists, it is typically the case that any specific model of the political process will affect the policy outcome resulting from a particular distribution of voter preferences. It is worthwhile, then, to attempt to draw broad conclusions regarding the class of model in question and the resulting solution(s). In line with this goal, McKelvey (1979) highlights the implications of global intransitivity for the power of the agenda setter while Caplin and Nalebuff (1988) show that if voters have spherical preferences[2] and the density of voters' bliss points is concave[3] then a 67% super-majority rule for changing the status quo ensures an unbeatable policy choice. These and many other contributions draw important conclusions regarding the relationship of the class of models considered (sincere voting models with an agenda setter in the former case and sincere voting with super-majority rules in the latter, for example) and the properties of solutions.

This paper aims to expand this literature by analyzing a new min-max solution to multidimensional social choice problems. The *Defensive Optimum* (DO) is defined to be the point that minimizes the distance to the most distant alternative choice that can beat the DO in pair-wise voting. For example, suppose a Condorcet winner exists for some distribution of voter preferences. The only policy that can beat a Condorcet winner is, by definition, that same Condorcet winner. Therefore the distance from a Condorcet winner to the most distant point that can beat it in pair-wise voting is zero. Zero being a lower bound on distances, the Condorcet winner is also therefore the defensive optimum. In general, Condorcet winners do not exist, and in these cases the resulting policy choice results from the particular model applied. Different voting institutions/rules will result in different policy choices, but a broad class of voting models will result in the defensive optimum as the resulting policy. Examples include models of political advertising, incumbency advantage, and interest group politics.

Models where the defensive optimum is the solution tend to have a defensive nature. In each case, there is some sort of protection for a policy/candidate when it is faced with an alternative policy/challenger. The protection may be exogenous or endogenous, and may come in the form of advertising, the ability to deliver benefits to voters, the nature of incumbency, etc. but essentially the defensive optimum is the most defensible policy choice. That said, it may arise in citizen-candidate models, random competition models or opportunistic

[1] There is a rich literature outlining what restrictions on preferences must be made to get some social choice. See Grandmont (1978), Kramer (1973), and Rubinstein (1979) for examples.

[2] Spherical preferences are those such that the actor's preference for a choice depends only on the Euclidean distance of that choice from some central bliss point or most preferred policy.

[3] This assumption is relaxed to log-concave densities and beyond in Caplin and Nalebuff (1991).

candidate models: while all share a defensive component, there are a wide variety of modelling approaches and specific contexts for which the DO is the solution.

The defensive optimum also exhibits the appealing feature that it always exists and is unique. Many solutions in the world of multidimensional voting, like the aforementioned Condorcet Winner, do not always exist, and those that do are often not unique (as with citizen-candidate models). Existence and uniqueness are an unusual pair in this literature, and that the defensive optimum possesses both qualities while remaining widely applicable is attractive. The defensive optimum also is continuous in changes in the voter density: small changes in the distribution of voter preferences do not have large affects on the resulting social choice, so standard empirical approaches using models with the defensive optimum as the solution will be well founded structurally. These features will allow political economists to design models resulting in the DO as their solution and rest assured that their models have unique solutions that behave "nicely".

To get a feel for the nature of the defensive optimum, I analyze the relationship between the distribution of voter preferences and the location of the DO. To ground the analysis in a recognizable framework, I consider a population consisting of two sub-populations, a majority and a minority. Each population has members with policy preferences covering the entire spectrum of preferences, so this division can be done with any population and is approximately without loss of generality.[4] I then adjust the characteristics of the minority population and analyze the resulting movement of the DO. I find that when the minority group is made to have less concentrated preferences (akin to adjusting the preference distribution via a mean preserving spread) the DO moves away from the center of the minority group and toward the center of the majority, in a way to be made precise later. The minority can be said to have less influence over policy. I also show that more distant minorities exert greater absolute influence: as the distribution is changed so that its center moves away from the center of the majority, the DO moves with it. I finally show that larger minorities exert greater influence (as one would expect) but that influence decreases less than one for one with size.

Section 2 shows that the defensive optimum exists, is unique and coincides with Condorcet winners if they exist. It also highlights that the DO is continuous with changes in the voter preference density and that it is always located on planes about which said density is symmetric. Section 3 relates the position of the defensive optimum to the size, location and concentration of a minority group relative to the larger population. Groups more able to shift the DO in their favor are defined to be more influential. Quantitative results involving

[4] "Approximately without loss of generality" means that any distribution with support S_1 can be approximated arbitrarily well as a distribution over support $S_2 \subseteq S_1$. Therefore any division of a density into two sub-densities can be arbitrarily well approximated as a division into two sub-densities that share common support.

the influence of very small minorities are presented in Section 4. The implications of the above for bloc voting are discussed in Section 5 and the differences between predictions of this model and the median voter theorem are discussed in Section 6. In Section 7 I describe three models in the areas of interest group politics, political advertising and pork, and incumbency advantage and describe how the DO is the result in each. I solve the model of interest group politics more precisely than the others, but the methods from that solution are readily applicable to the other models. Section 8 concludes. Some simple proofs will be in the text if they prove useful in the exposition; otherwise proofs are contained in the appendix.

2 The Defensive Optimum

The solution to models with multidimensional preferences depends critically on the model of voters and political institutions employed by the researcher. It is often desirable to describe a particular solution concept that may apply in a wide variety of modelling contexts. The solution concept can be analyzed so that a researcher with a particular model in mind can assume properties like existence or uniqueness without having to prove them herself. A Condorcet winner is an example of such a broad concept, as is the defensive optimum described below. Section 7 outlines some examples where the defensive optimum is the resulting policy choice in more detail, but the analysis in this section does not rely on any particular model.

To begin, we should define the modeling assumptions that are necessary to establish the properties of the defensive optimum described below. The defensive optimum is defined to be the location in the policy space that minimizes the distance to the farthest alternative policy choice that would beat the DO in sincere pair-wise voting. That is, suppose we have some candidate policy. As is well known, in a multidimensional policy space there are typically many alternative policies that would beat the candidate in pair-wise voting (i.e. a Condorcet winner does not generally exist). Without delving into details of a model, we might imagine that a "stronger" candidate would be one for whom alternative winning policies would be closer to the candidate. If this is indeed the appropriate concept of strength, then the strongest policy is the one whose most distant alternative winning policy would be closest. This is our definition of the defensive optimum.

In order to derive properties of the defensive optimum, we must describe a few things. First, how does our voting mechanism decide which policy would "win" in a pair-wise election? Second, what is the set of alternative policies available? Third, what is our concept of "distance", as the word is used above? To answer these questions we outline the four assumptions below that define the setting. Let x be a vector describing a policy.

- A1 Voters have spherical preferences. That is, they have utility $U(x,y) = u(|x-y|)$ where u is decreasing, x is the voter's "bliss point", or most preferred policy, and y is the eventual social choice.[5] The "distance" between two options is therefore Euclidian distance.

A2 The set of available policies, $S \subset \mathbb{R}^N$, is convex, closed and bounded.

A3 .$f(x)$ is the density of bliss points in the population of voters where $0 < f(x) < \infty$, $\forall x \in S \subset \mathbb{R}^N$. $f(x)$ is differentiable everywhere on the interior of S.

A4 x would win a pair-wise election against y iff $\int_{|x-z|<|y-z|} f(z)dz > \int_{|x-z|>|y-z|} f(z)dz$.

A2 and A3 are stronger than necessary but make the proofs simpler. In fact, A1 could be replaced by any single peaked utility function and the defensive optimum would remain generically (instead of generally) unique,[6] but at the expense of considerable complication. Before we can formally define the defensive optimum, we must define terms that should aid in the exposition of both the definition and some following results. For the purposes of sincere voting models with spherical preferences, a density of voters can be thought of as a collection of hyperplanes, each of which divides the density in half. This is a result of the method we use to find who would win in pair-wise voting described in assumption A4.

How do we find the set of alternatives that can beat some choice x in an election? Choose some direction $\theta \in [0, \pi)^{N-1}$ and consider some point z at angle θ from x. Construct a line between x and z and consider the hyperplane orthogonal to this line halfway between the two. If more mass of f lies on the side of this hyperplane closer to x then x wins; otherwise z wins. The reason is that all points on the side of this hyperplane containing x are closer to x using Euclidean distance as our measure of distance. Those voters would prefer to vote for x versus z. The mass of voters on the side of this hyperplane containing x defines the "number" of voters who vote for x. If it is bigger than the number who vote for y then x wins.

Next, note that if z beats x in an election, then all points on the line connecting z and x would also beat x in an election. Now consider the set of points in direction θ from x, each of which beats x in an election and call this set $w(\theta, x)$. Let $x'(\theta, x) = \sup_{s \in w(\theta,x)} |s - x|$. Then $x'(\theta, x)$ is the most distant point in direction θ from x that can beat x in pair-wise voting.

We want to understand for any x how to describe the set $W(x)$ that contains only alternatives that can beat x in pair-wise voting. This is done by finding, for every direction vector $\theta \in [0, \pi)^{N-1}$, the hyperplane orthogonal to θ that divides the mass of f in half as shown in figure 1. There will be one, and only one (a result that holds because f is positive on S and S is convex) such hyperplane. $x'(\theta, x)$ is the reflection of x across this hyperplane (if x lies on

[5] Note that spherical preference satisfy "single peakedness," but over multiple dimensions of policy. That is, all voters' utilities are decreasing in all directions from their most preferred candidate. Also note that spherical preferences have been used in many models of social choice including Caplin and Nalebuff (1989 & 1991) and Grandmont (1978).

[6] Proof available upon request.

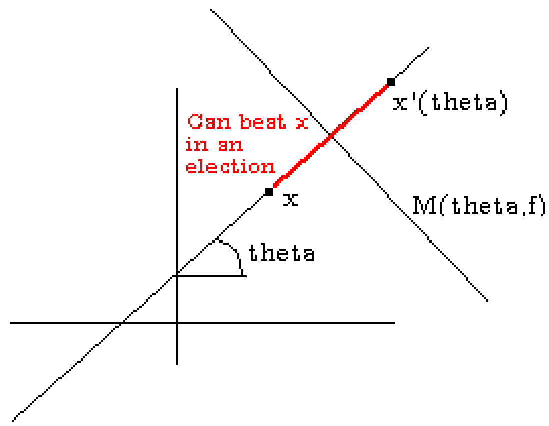

Figure 1: For a given policy x, we find the alternatives that can beat x in pairwise voting by, for ever direction θ, finding the reflection of x, denoted $x'(\theta, x)$, across the f-median associated with θ. The points on the segment connecting x and x' can beat x in pairwise voting.

the hyperplane then $x'(\theta, x) = x$). Now, for any x, the set of alternatives in direction θ that can beat x in pair-wise voting is the set that lies on the line segment between x and $x'(\theta, x)$. In figure 1, as x moves to the southwest along the line in direction θ, $x'(\theta, x)$ moves to the northeast. As x gets farther from the hyperplane labeled $M(\theta, f)$, more alternatives in direction θ can beat x in pair-wise voting.

These concepts will arise so often in this paper that we need better names and symbols:

Definition 1 *An f-median is a hyperplane that divides the mass of voters in half. The symbol $M(\theta, f)$ denotes an f-median of the density f orthogonal to a direction vector $\theta \in [0, \pi)^{N-1}$.*

Definition 2 *The Fundamental Function $B(f, x, \theta)$ identifies the distance from any policy x to the most distant f-median in a direction θ:*

$$B(f, x, \theta) = \|x - M(\theta, f)\| \tag{1}$$

$B(f, x, \theta)$ is half the distance to the most distant alternative to x in direction θ that can beat x in an election.

Note that the distance metric $\|\bullet\|$ here represents the distance between a point and a line. If x is a point and L is a line, $\|\bullet\|$ solves

$$\|x - L\| = \min_{z \in L} |x - z|$$

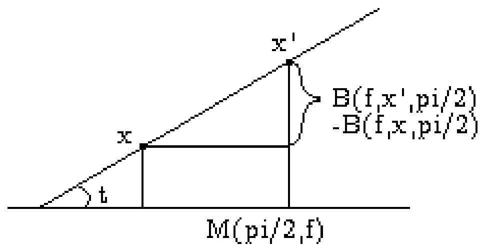

Figure 2: In $\mathbb{R}^2$, as x moves to x', we see that the distance to the f-median associated with $\theta = \frac{\pi}{2}$ changes by $|x - x'| \sin t$ where t is the angle between the f-median and the line containing x and x'. This implies that $\frac{\partial B}{\partial x} = \sin t$ so B is differentiable.

Definition 3 *The Defensive Optimum $x^*(\theta)$ is the solution to the following min-max statement*

$$x^*(\theta) = \arg\min_x \max_{\theta \in [0,2\pi)^{N-1}} |2 \times B(f, x, \theta)| \tag{2}$$

Because $B(f, x, \theta) = -B(f, x, \theta - \pi)$ and because $\max_s 2f(s) = \max_s f(s)$, we can write the defensive optimum more simply as:

$$x^*(\theta) = \arg\min_x \max_{\theta \in [0,\pi)^{N-1}} |B(f, x, \theta)| \tag{3}$$

The fundamental function is shown in the appendix to have some very nice properties. Specifically, it is continuous in f and differentiable in x and θ. The differentiability in x follows immediately: as shown in figure 2 for the case of $\mathbb{R}^2$, the derivative of $B(f, x, \theta)$ with respect to x as x moves in direction D is $\sin t$ where t is the angle between the f-median $M(\theta, f)$ and the line through x in direction D. This argument clearly extends to $\mathbb{R}^N,\ N > 1$.

The differentiability in θ follows from our assumption that the voter density is non-atomistic, assumption A3. Essentially, since the $\int$ operator is continuous for bounded functions, $B(f, x, \theta)$ is continuous in θ. Differentiability is trickier, but follows similarly. Continuity in f is more involved but the intuition is straightforward. Small changes in the density of voters yield small changes in the locations of the f-medians. This clearly implies that the distance from any particular point to any particular f-median will also be small so we have continuity.

The following results establish useful properties of the defensive optimum.

Proposition 1 *The defensive optimum exists*

Proposition 2 *The defensive optimum is unique*

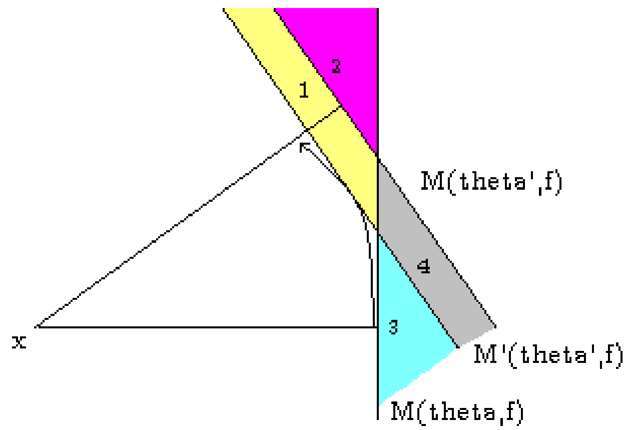

Figure 3: $B(f, x, \theta') - B(f, x, \theta)$ is the distance between lines $M'(\theta, f)$ and $M(\theta, f)$. This distance must be such that the mass of area 2 equals the mass of areas 1, 3 and 4. As $\theta' \to \theta$, the areas of 2 and 3 go to 0. Therefore $B(f, x, \theta') - B(f, x, \theta)$ must go to 0. This establishes continuity, but differentiability is a little extra work.

Proposition 3 *The defensive optimum is continuous in the voter density f*

These three properties make the defensive optimum a particularly attractive solution in multidimensional voting problems. Solutions to these problems often suffer from potential nonexistence, and when existence is ensured there are typically a multiplicity of equilibria. That we can achieve both existence and uniqueness is unusual and useful: theorists creating models with the DO as the solution can rest assured that existence and uniqueness hold. We now establish a simple result that is intuitively appealing as well as useful in our later analysis.

Definition 4 *A density f is laterally symmetric across a hyperplane A if f is symmetric with respect to A*

Proposition 4 *If a density f is laterally symmetric across some set of hyper planes (each of dimension $N-1$) denoted A^*, then the defensive optimum $x^*(f)$ lies on the intersection of the hyperplanes contained in A^**

Proof. If we have some DO $x^*(f)$ off of some $A \in A^*$, then we would have another one reflected across A. Since $x^*(f)$ is unique, this is a contradiction. ■

The preceding proposition shows that the density of voters can be positive over many dimensions, but as long as the voters are spread symmetrically across some hyperplane, we can be assured that the DO lies on that hyperplane. For example, suppose the voter density is uniform over a hypercube centered at the origin. Let a point in this space be defined by $x = (x_1, x_2, ..., x_N)$. Then the voter density is symmetric across hyperplanes identified with $x_i = 0$ for any i. Therefore the defensive optimum lies at the intersection of these hyperplanes, and this intersection is, uniquely, the origin. There is no Condorcet winner for this voter density, but the defensive optimum picks the most "natural" location for a policy choice. Other examples of voter densities where the DO is the origin would be:

1. Uniformly distributed voters over a regular N-polygon centered at the origin
2. Uniformly distributed voters over a barbell centered at the origin
3. Multivariate normally distributed voters with the mean at the origin

The DO also has the attractive feature that it equals the Condorcet winner if one exists.

Proposition 5 *If a Condorcet Winner exists, then $x^*(f)$ is the Condorcet Winner*

Proof. If a Condorcet Winner exists then the distance to the most distant alternative policy that can beat it in pair-wise voting is zero. Because distances must be weakly positive, the Condorcet winner is necessarily the defensive optimum. ■

Plott (1967) and Davis, DeGroot and Hinich (1972) showed that Condorcet winners exist if and only if all f-medians intersect at a single point. The Condorcet Winner is accordingly not a particularly useful tool for evaluating comparative statics when policy is multi-dimensional, but it is important that in the rare instances in which it is useful our predictions align.

The above propositions establish the utility in working with this framework. Unlike many models of electoral competition in multiple dimensions we are assured a generally unique outcome that always exists, and this outcome corresponds to that predicted by Condorcet when possible. In order to aid future researchers who make use of the defensive optimum for their models, it is worth describing how the location of the DO relates to properties of the underlying distribution of voter preferences.

3 The Defensive Optimum and Minority Influence

In attempting to relate characteristics of a population to the location of the defensive optimum, there are two approaches one could take. First, one could define population distributions according to moments (mean, variance, skew, etc.), shapes (circles, squares, barbells, etc.) or other similar properties. The laterally symmetric median voter theorem from proposition 4, however, makes many shape comparisons straightforward (most standard shapes are symmetric across some hyperplanes, and the defensive optimum will therefore lie at the "middle" of the shape) and the analysis using moments alone is not particularly productive without first constraining the general shape of the distribution.

Second, one can split distributions into components and evaluate how the location depends on the size and shape of each component. This method has the additional benefit of having a practical interpretation: what groups within a population are able to wield more influence over policy? As we change the shape, size or location of a sub-population, the location of the defensive optimum will change as well. We can therefore identify which "minority" groups will wield the greatest influence over policy, an important question on its own as well as a convenient way to describe the location of the DO in relation to population characteristics.

The following three definitions make explicit the meaning of size, concentration and location. When we say, for example, that more concentrated minority groups exert more influence over policy, we need to make sure that these groups are the same in other important ways.

The size of a group is simply the integral under its voter density.

The location is more difficult: in this section both majority and minority densities are assumed to share full support over S and are in that sense "in the same location." However, if the bulk of voters in each group are in different areas of the support then, for practical purposes, the groups are "in different locations."

Definition 5 *Two densities f and f' are defined to be equivalently located their f-medians are identical; i.e. if $M(\theta, f) = M(\theta, f^{'}), \forall\theta$ or, equivalently, $B(f, x, \theta) = B(f^{'}, x, \theta)$.*

Example 1 *If f and f' are both uncorrelated normals with differing variances in each direction, then f and f' are equivalently located.*

Example 2 *If f and f' are uniform regular polyhedrons with sides of differing length, both with center at the origin, then they are equivalently located.*

Definition 6 *Two densities f and f' are differentiated if $\exists A$ where:*

1. *$M(\theta, f)$ is weakly closer to A than $M(\theta, f')$ for all θ*
2. *$M(\theta, f)$ is strictly closer to A than $M(\theta, f')$ for some θ*

In a slight abuse of language, we will also sometimes say that, if the above holds, f is closer than f' to A and we will also describe f and f' as differentiated with respect to A.

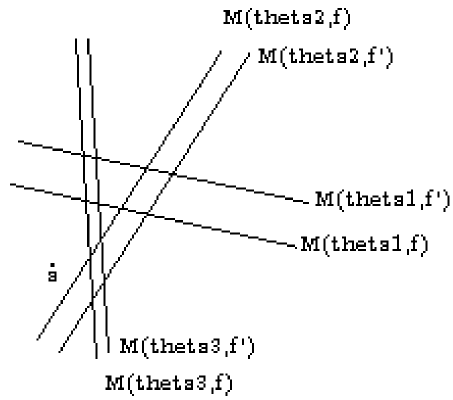

Figure 4: f and f' are differentiated in this figure. Each f-median of density f is closer to a than the associated f-median for the density f'. Only three f-medians are plotted here and to properly check that f and f' are differentiated we would need to check all f-medians.

Example 3 *Suppose f' is just f translated in any direction. Then f and f' are differentiated.*

Once we know the locations of the f-medians, we could say that group H is, for example, "to the left" of group K if all f-medians for group H are on the "left" side of the associated f-medians for group K. Figure 4 shows the case of differentiated densities in $\mathbb{R}^2$. Each f-median of f shown is closer to a than the associated f-median for f'. To properly check that f and f' are differentiated we would need to check that this is true for all f-medians, but the meaning of differentiation should be clear from the figure.

It is not the case that any two densities are either equivalently located or differentiated: these classifications are not mutually exhaustive. However, we need to refer to location for two reasons: First, to hold location constant as we vary the size or concentration of a density and second, to evaluate the effect of distance between a majority and minority on the influence of the minority. The first use requires various minority densities to be equivalently located and the second requires them to be differentiated.

To define concentration, denote a CDF in direction θ starting at $M(\theta, f)$ in direction $y \in \{+, -\}$ to be $\Delta^y(f, \theta, d)$. That is, let $C(s)$ be a plane parallel to $M(\theta, f)$ at distance s from M. Both C and M are orthogonal to the vector at angle θ (by definition, because they are parallel). There are two planes that are distance s from M: one which is "in the direction" θ from M (which we call $(+)$) and one that is on the opposite side of M ("in the direction" $-\theta$ from M

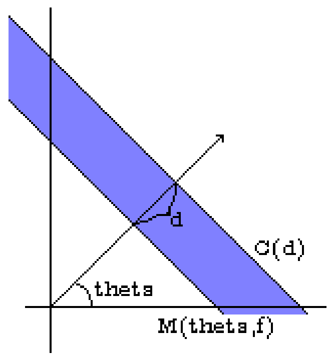

Figure 5: $\Delta^+(f, \theta, d)$ is the integral of f over the shaded area between $M(\theta, f)$ and a parallel hyperplane distance d from M in the same direction as θ. In this figure, $\Delta^-(f, \theta, d)$ would be the integral of f between M and the reflection of $C(d)$ across M.

which we call (-)). Let C be in the (+) direction. Then

$$\Delta^y(f, \theta, d) = \int_0^d \int_{C(s)} f(x) dx ds$$

is the volume of the area between M and $C(d)$. In figure 5, $\Delta^+(f, \theta, d)$ is the integral of f over the shaded area.

Definition 7 *A density f is less concentrated than another density f' if:*

$$\Delta^y(f', \theta, d) \geq \Delta^y(f, \theta, d), \forall \theta, d, y$$

Example 4 *If f and f' are both uncorrelated normals and the variance of f in each direction is less than the variance of f' in the same direction, then f is more concentrated than f'.*

This notion of concentration is akin to the concept of a mean preserving spread. This definition will allow a discussion of the effect of the concentration of members' preferences on influence.

For several examples in this paper we use voter densities that we call radially symmetric. In the prior literature this term has taken one of two meanings. Some have defined radial symmetry as the property of a density that there exists some point A for which the integral along any line through A is equal on each side of A. We define the property according to the second convention:

Definition 8 *A density f is said to be radially symmetric about a point A if* $f(d,\theta) = f(d)\forall\theta$ *(where locations are expressed in radial coordinates, choosing A to be the origin, and* θ *is a* $N-1$ *direction vector).*

Alternately stated, a density is radially symmetric about A if its value at a point x depends only on the distance from x to A. This is "radial" because f is constant on spheres centered at A. We will temporarily restrict the types of changes in voter densities we will consider for the purpose of getting analytical results. When we refer to a minority becoming larger, more concentrated or more distant, we will require the changing voter densities to have a characteristic called the movement property:

Definition 9 *A set of densities* Ω *satisfies the movement property if for any* $f, f' \in \Omega$ *and* $A \in \mathbb{R}^N$ *where* f *is closer than* f' *to A:*

$$|x^*(f) - A| \leq |x^*(f') - A|$$

That is, Ω *satisfies the movement property if, as densities in* Ω *get closer to A, their DOs get closer to A.*

Example 5 *Let* f, f' *be radially symmetric densities symmetric about points A and B respectively. Let* $\Omega = \{h : h = (1-w)f + wg, w \in [0,1]\}$*. Then* Ω *has the movement property. Each density* $h \in \Omega$ *is a weighted average of two radially symmetric densities. Each is therefore laterally symmetric across planes containing the line connecting A and B and the DO therefore lies on that line. As the weight shifts increasingly on* f' *from* f*, the DO moves down the line toward B.*

The purpose of this section is to understand the effect of minority size, concentration and location on influence, so we need a definition of influence. Normally we think of a group as being more influential when policy is affected more by its presence. The most natural way to frame this is to compare the actual policy outcome to what it would have been had the group not existed. Let a population h be comprised of two sub-populations with densities f and g.

Definition 10 *Let there be two sub-populations f and g, each a component of* $h = (1-w)f + wg, w \in [0,1]$

1. *The absolute influence of g is equal to*

$$|x^*(f) - x^*(h)| \tag{4}$$

2. *The relative influence of g is equal to*

$$\frac{|x^*(f) - x^*(h)|}{|x^*(f) - x^*(g)|} \tag{5}$$

The absolute influence of a group is how much the group's existence shifts policy in absolute terms. This clearly depends on units and is therefore most useful in comparing the influence of two minorities at differing distances from a fixed majority. The ratio of the absolute influence of the two is then unitless. The relative influence is interpretable as the distance from the DO of the overall population to what would be the DO if f were the only group in the population, divided by the distance from the DO of f to the DO of g separately. This is unit-less, unlike the absolute influence.

These definitions have perhaps been slow going, but the end goal is to show that larger, more concentrated and more politically distant minorities exert greater influence on policy. We now have the definitions in place to precisely understand the meaning of "concentration" and "political distance." In order to understand the effect of concentration on the location of the DO, we want to compare two minorities of equal size and location fixing the majority group.

Proposition 6 *More concentrated minorities are more influential. Suppose we have f, g, g' where g and g' are equivalently located and g is more concentrated than g'. Suppose that f and g are differentiated with respect to point $x^*(g)$.*[7] *Let $h = (1-w)f + wg$ and $h' = (1-w)f + wg', w \in [0,1]$ and let the set $\{h, h'\}$ have the movement property. Then minority g is more absolutely and relatively influential than minority g':*

$$|x^*(f) - x^*(h)| \geq |x^*(f) - x^*(h')| \tag{6}$$

Roughly speaking this means that if we have two potential voter densities with the same location but differing concentrations and if these densities are both parts of the same larger population, the DO of the overall population will be closer to the would-be DO in the more concentrated density. The argument is as follows: The combined population with the more concentrated minority density will have f-medians closer to the would-be DO of the minority density. Because the set of densities we consider has the movement property, if the f-medians are closer to the would-be DO then the true DO must also be closer to it.

Corollary 1 *Larger minorities are more absolutely and relatively influential. If $w_1 > w_2$, and f and g are defined as above, $h_1 = (1-w_1)f + w_1 g$ and $h_2 = (1-w_2)f + w_2 g$, then*

$$|x^*(f) - x^*(h_1)| \geq |x^*(f) - x^*(h_2)| \tag{7}$$

This result simply shows that larger minorities will exert more influence than smaller ones so long as our assumptions are satisfied. While this seems obvious,

[7] If f and g are differentiated with respect to point $x^*(g)$, then f and g' are as well. The f-medians of g and g' are the same, and differentiation is a property of the f-medians. Further, $x^*(g) = x^*(g')$ because x^* depends only on the f-medians of a density and g and g' share f-medians.

relaxing the assumption that our class of functions has the movement property allows counterexamples where increasing size or concentration of a minority can reduce its influence.

Finally we need to discuss the effect of population location on policy.

Proposition 7 *More distant minorities are more absolutely influential. Suppose we have f, g, g' where g is closer than g' to $x^*(f)$ and f and g are differentiated with respect to $x^*(f)$. Let $h = (1-w)f + wg$ and $h' = (1-w)f + wg', w \in [0,1]$ and let the set $\{h, h'\}$ have the movement property. Then minority g' is more absolutely influential than minority g:*

The preceding results provide comparative statics for the location of the defensive optimum as we vary the population density. As a sub-population becomes more concentrated and larger, it exerts greater absolute and relative influence. As it is more distant from the rest of the population, it is more absolutely influential. We cannot sign the change in its relative influence as it is more distant. These results on their own are useful and correspond to intuition. To get more precise predictions, it is necessary to make stronger assumptions on the voter density. The implications of one such restriction are fleshed out below.

4 Influence of an Atom Minority with a Radially Symmetric Majority

We have seen thus far that larger, more concentrated and more distant minorities exert greater influence over policy. For the rest of the paper we restrict attention to atomistic minorities: this allows precise calculations of the effects of size and distance on absolute and relative influence. Because all atoms are, by definition, infinitely concentrated, we will not be able to get precise calculations of the effect of minority concentration on influence. We will, however, be able to analyze the effect of majority concentration on minority influence, which is presumably analogous.

In this section we show that more distant minorities have more absolute influence and less relative influence and that very small minorities have disproportionate power.[8] We also will be able to investigate complementarity in minority group properties. Distance and concentration are compliments: the effect on minority influence of a reduction in majority group concentration is greater for more fringe minority groups. Distance and size are also compliments: the effect of an increase in size on influence is greater for more fringe groups. Concentration and size are not complementary, however, as a reduction in majority concentration reduces the effect of an increase in minority population on minority influence.

[8] We showed in section 3 that larger minority groups are more influential, but did not address *how much* more.

Propositions 1 and 2 assume that the voter density is not atomistic and this assumption is violated here, but by restricting attention to radially symmetric majority densities uniqueness and existence are maintained.[9] For simplicity of exposition, let the following framework be called our "simple model."

B1 f is a radially symmetric density of mass $1-\varepsilon$ centered at point A

B2 g is an atom of size ε located at B

B3 The number of dimensions is $N \geq 2$

Let $\Delta(d) = \Delta^y(f, \theta, d)$. Because f is radially symmetric, the area under f in any direction from A is equal, so we can drop the θ and y. We also leave the dependence of Δ on f implicit so that we can take an inverse.

Proposition 8 *In our simple model, the defensive optimum is located on the line connecting A and B. The absolute influence of the minority is*

$$y = \frac{2\Delta^{-1}(\nu)D}{D + \Delta^{-1}(\nu)} \tag{8}$$

so long as $\varepsilon \leq \frac{2\Delta(D)}{1-2\Delta(D)}$, *where D is the distance from A to B and* $\nu = \frac{1}{2}\frac{\varepsilon}{1-\varepsilon}$. *Otherwise, relative influence* $\frac{y}{D} = 1$ *because the DO will lie on point B.*

This equation will allow us to easily calculate absolute and relative influence of minorities as a function of majority concentration (the value of Δ^{-1}), minority size (ν is monotonic in ε and Δ^{-1} is monotonic in ν) and the location of the minority relative to the majority: y is the absolute influence of group g and $\frac{y}{D} = \frac{2\Delta^{-1}(\nu)}{D+\Delta^{-1}(\nu)}$ is the relative influence.

Proposition 9 *In our simple model, the absolute influence of the minority is increasing in distance from the majority but the relative influence is decreasing in this distance.*

Proof.

$$\frac{\partial y}{\partial D} = \frac{2\Delta^{-1}(\nu)^2}{(D + \Delta^{-1}(\nu))^2} > 0 \tag{9}$$

so the absolute influence of g is increasing in D. The relative influence is given by

$$\frac{y}{D} = \frac{2\Delta^{-1}(\nu)}{D + \Delta^{-1}(\nu)} \tag{10}$$

and is clearly decreasing in D:

$$\frac{\partial}{\partial D}\frac{y}{D} = -(\frac{2\Delta^{-1}(\nu)}{D + \Delta^{-1}(\nu)})^2 < 0 \tag{11}$$

[9] Furthermore, atomistic densities can be approximated arbitrarily well with non-atomistic ones that satisfy all of our assumptions, and because the DO is continuous in voter density f, the approximation should yield identical results in the limit. Assuming non-atomistic densities was most useful for making the exposition simpler.

■

This shows that comparing two otherwise identical potential minority groups C and D where C's views are more fringe (i.e. more distant from those of the majority), C's existence will shift policy more than the D's, but C may feel more dissatisfied with the final policy chosen.

The relationship between minority size and influence is monotonic, but we might consider proportionate influence (influence per member) important. The ratio of relative influence to minority size is $\frac{1}{\varepsilon}\frac{2\Delta^{-1}(\nu)}{D+\Delta^{-1}(\nu)} = \frac{1-2\nu}{2\nu}\frac{2\Delta^{-1}(\nu)}{D+\Delta^{-1}(\nu)}$. $\frac{1-2\nu}{2\nu}$ is decreasing in ν (because $0 < \nu < \frac{1}{2}$) but $\frac{2\Delta^{-1}(\nu)}{D+\Delta^{-1}(\nu)}$ is increasing in ν, and the rate of increase depends on the shape of Δ which can take many forms. The proportionate influence of minorities depends on the properties of the majority in a complex way.[10] If we focus, however, on the relative influence of very small minorities we can get some traction. How much influence does a vanishingly small group exert over policy?

Proposition 10 *In our simple model, the proportionate absolute influence of a vanishingly small minority is π., the proportionate relative influence is $\frac{\pi}{D}$ and the proportionate marginal influence is $-\frac{4\pi^2}{D^2}$, where $\pi = 3.14...$*

The above proposition establishes that the absolute influence of a vanishingly small minority is π regardless of the attributes of that minority.[11] If we think of these minorities individuals, every person in society has exactly equal pull, regardless of the distance to the eventual policy choice. Further, this number is greater than 1, implying a disproportionate influence over policy. Just as it is possible to have many pivotal voters in some models, here every voter has disproportionate power.[12]

There are three complementarity results, two of which compound the effectiveness of certain types of minority.

Proposition 11 *In our simple model, majority preference dispersion and distance from the minority to the majority are complementary*

Proof. The mixed partial derivative of absolute influence with respect to distance and majority concentration is

$$\begin{aligned}\frac{\partial^2 y}{\partial D \partial \Delta^{-1}(\nu)} &= \frac{\partial y}{\partial \Delta^{-1}(\nu)} \frac{2\Delta^{-1}(\nu)^2}{(D+\Delta^{-1}(\nu))^2} \\ &= \frac{4D\Delta^{-1}(\nu)}{(D+\Delta^{-1}(\nu))^2} > 0\end{aligned}$$

■

[10] The effect of a small increase in minority size shifts policy more or less depending on the height of f at certain key f-medians. The interpretation is that if there are few swing voters then increases in minority size affect policy more than when there are more swing voters.

[11] In fact, this would be true for non-atom minorities so long as a circle of radius <D centered at B contains all the mass of g.

[12] This means that at the margin, all voters have disproportionate power. Clearly on average each voter cannot have disproportionate power.

Proposition 12 *In our simple model, distance from the majority and size of the minority are complimentary*

Proof. The mixed partial derivative of absolute influence with respect to distance and minority size is

$$\begin{aligned}\frac{\partial^2 y}{\partial D \partial \varepsilon} &= \frac{\partial y}{\partial \varepsilon} \frac{2\Delta^{-1}(\nu)^2}{(D + \Delta^{-1}(\nu))^2} \\ &= \frac{\partial \nu}{\partial \varepsilon} \frac{4D^2 \Delta^{-1}(\nu)(\frac{\partial}{\partial \nu}\Delta^{-1}(\nu))}{(D + \Delta^{-1}(\nu))^4} > 0\end{aligned}$$

■

Proposition 13 *In our simple model, majority concentration and size of the minority are complimentary*

Proof. The mixed partial derivative of absolute influence with respect to majority concentration and minority size is

$$\begin{aligned}\frac{\partial^2 y}{\partial \varepsilon \partial \Delta^{-1}(\nu)} &= \frac{\partial}{\partial \varepsilon} \frac{2D^2}{(D + \Delta^{-1}(\nu))^2} \\ &= -\frac{2(\frac{\partial}{\partial \nu}\Delta^{-1}(\nu))\frac{\partial \nu}{\partial \varepsilon}}{(D + \Delta^{-1}(\nu))} < 0\end{aligned}$$

■

The preceding three propositions highlight that distance interacts positively with both size and minority dispersion: as minority groups are more distant from the majority, the effect of increasing size and reduced majority concentration reinforce the benefit of being distant. Size and majority dispersion are not complementary, however, and each negatively impacts the effect of the other.

While no data will be presented in this paper to support these results, the facts presented in this section seem to square with casual observation. Tiny fringe groups seem to have a surprising amount of influence in policy making- the eventual outcome tends to be centrist because the mass of voters is centrist, but it is often striking the degree to which policy is shifted away from the center by these groups. Many authors have suggested that campaign contributions, lobbying or other institutional protections are behind these observations, but this paper suggests that those explanations, while likely important, need not tell the whole story.

5 Implications for bloc voting

Throughout the paper we have assumed that voters vote sincerely. Because there is a continuum of voters, each voter does not influence the outcome and therefore is indifferent as to her vote. It seems natural under these circumstances to assume sincere voting: After all, beyond the potential to affect the outcome of an election, the only reason to vote is seemingly some personal "good feeling" that stems directly from the act of voting. There might, however, be some other direct utility benefit to voting in a particular way. For example, voting in line with friends, family, co-workers or co-religionists might generate good feeling. It is long established that voters who group themselves in blocs are more effective at influencing policy. While we do not tackle the subject in full here, the above results have some implications for bloc voting.

First, the importance of concentration makes clear why voting as a bloc can benefit the group as a whole. A small minority will have only a small influence on policy. By committing to vote as a bloc they can shift policy more in their favor. Second, peer pressure to vote increases the size of the minority and therefore its influence. Third, groups may influence members to vote "more extremely" than they otherwise would. If a bloc of voters could commit to voting *as though* they were more fringe, they could shift policy more in their favor. This behavior does, perhaps, seem to have some real world examples. Both sides of many policy debates will claim to have fringe beliefs and will adamantly vote against a moderate agenda. In both sides of the abortion, civil liberties and gun debates, one could argue that the beliefs of activists on either side are likely not as extreme as their public statements (and exhortations of followers) suggest. The implications above suggest that we should expect no less.

6 Comparisons to the Median Voter Unidimensional Case

It is useful to compare the results we have obtained for minority influence to the unidimensional case. Here, the median voter theorem will apply, so our analysis can simply compare to that. Suppose we have a majority density f with full support and DO A and a minority atom g at location $B > A$ of size ε. Letting Δ be defined as before, and letting $D = B - A$ with $\Delta(D) = d$, the location of the median voter is given by

$$y = \Delta^{-1}(\nu) \tag{12}$$

as long as $\nu \leq d$. Recall that with at least two dimensions we had $y = \frac{2\Delta^{-1}(\nu)D}{D+\Delta^{-1}(\nu)} = \frac{2D}{D+\Delta^{-1}(\nu)}\Delta^{-1}(\nu)$. Since $\Delta^{-1}(\nu) < D$ whenever $\nu \leq d$, we get $\frac{2D}{D+\Delta^{-1}(\nu)} > 1$. Therefore two dimensions give minorities more influence than one, but more than two yields no greater minority influence than two. The intuition is straightforward: in one dimension a policy only faces competition from alternatives on either side along the line, but in two it faces competition from off the line L connecting A to B. The largest competitive threats come from voters closer to B than the median along L so the DO moves toward B to compensate. Because these competitive threats come from all other dimensions equally and can be addressed through the same move toward B, adding more dimensions after the second does not affect our results.

This comparison suggests that minorities wield more influence when the policy space is thought of as multidimensional, even when the views of each group along the extra dimensions are the same. Beyond this, we see that our qualitative results are shared in the unidimensional case. When f is more concentrated, the minority has less influence and when ε is larger, the minority has greater influence. Distance does not affect absolute influence in the unidimensional case so long as the median voter remains between A and B. However, as a minority groups gets large, the DO eventually reaches B. When this occurs, an increase in minority size does not affect minority influence (influence is already complete as the median voter is part of the minority atom). In the unidimensional case, therefore, size and distance are again complementary, though only weakly so. Majority dispersion and minority size are complementary for the same reason: More disperse majorities yield the median voter reaching B more quickly as the minority size increases. The advantage (in terms of absolute influence) of being more distant is therefore increasing with majority dispersion.

7 Models with the Defensive Optimum as the Solution

To this point the concept of the defensive optimum has been defined and its properties analyzed, but there has been little discussion of why we should care. In this section I outline three models of electoral competition for which the defensive optimum will be the resulting policy choice. For the first model, concerning "disinterested interest groups," I define precisely the form of the game and provide a rough derivation of the policy outcome implied by the game. The other two models I present, concerning the interaction of political advertising and pork with incumbency advantage and the impact of entrenched incumbents, are not accompanied by a derivation of the solution to the model but it should be readily identifiable based on the analysis of disinterested interest groups. It should be noted that all of these examples have a flavor of "defensiveness" about them. Each shows that the defensive optimum is the most defensible policy in its associated circumstance; hence the name "defensive optimum".

7.1 Disinterested interest groups

The most common approach to understanding the importance of interest groups in the theoretical literature has been to assume that interest groups care about the same types of policies that are important to voters. These models cover a range of assumptions about the composition and behavior of these groups: Downs (1957) first formally introduced the importance of politicians' incentives on policy, though this was not in an interest group setting. More recently, Bernheim and Whinston (1986) and Grossman and Helpman (1994) analyzed interest group politics by allowing lobbies to bid directly to influence profit maximizing politicians. Gary Becker (1983) considered a finite number of homogeneous groups of individuals who can apply costly pressure to politicians while Besley and Coate (1997b) embed exogenously set lobbies within a citizen-candidate framework.

All of these and other analyses, so far as I know, consider the situation where interest groups and voters have preferences along the same dimensions. For example, in presidential politics, the primary interest groups may focus on issues in the realms of religion, labor, business, taxation or other areas of real concern to most voters. In small elections, however, it is unclear that primary financial donors are interested in these issues. Instead, a donor might want a zoning change, an extra police officer assigned to a particular block, a government backed loan or other political pork. Voters may primarily interested in broader issues like school funding, property taxes, etc. Even in larger elections, like those for the US House of Representatives, there are many donors who expect spending or targeted tax cuts that are of little interest to voters. Sugar producers, for example, are large donors to political campaigns with the implicit agreement that their candidates maintain high sugar tariffs. The electorate appears indifferent regarding sugar tariffs. While voters periodically do complain about political pork– the hoopla surrounding the 2005 Transportation Equity Act, which famously allocated $223 million to build a "Bridge to Nowhere" in Alaska, is a rare example– the degree of their interest is typically significantly less than the degree of interest from donors.

I model the two groups here as being indifferent to policy dimensions that are of interest to the other. Because, for the most part, allocation of financial benefits to one donor does not preclude allocation of benefits to another, donors can get their candidate elected at minimum cost by joining together. In deciding on a candidate to support, they choose the one who is most able to deliver the desired benefits and who is most electable. The former may be achieved by supporting incumbents: leadership roles in important Congressional committees like Appropriations and Ways and Means are assigned by seniority and are important for delivering targeted benefits (spending in the former and taxation in the latter) to a home district.[13] The latter may be achieved by supporting a candidate whose policies are defined by the defensive optimum, analyzed above. In a dynamic setting where supported candidates periodically must be reelected, the two coincide.

In the following model voters will elect citizen-candidates, citizens who run for office with the goal of implementing their most preferred policies. Unlike the standard framework, one candidate will be selected to stand for office by a group of financial backers who will fund the candidate's campaign while other candidates must fund their own campaigns. These financial backers will have policy preferences along dimensions that are not considered relevant by voters, nor the candidate herself. I show that the financial backers' problem is identical

[13] More on this in the next example, on incumbency advantage.

to the min-max problem defining the defensive optimum and that the financial backers will therefore support a candidate whose MPP is the defensive optimum. This location minimizes the cost of getting their preferred candidate elected.

The game has four stages. In stage one, a group of financial backers chooses a citizen to support as a candidate and chooses a spending level c for the campaign. The next three stages follow the typical path for a citizen-candidate game. In stage two, citizens simultaneously choose to run for office; in stage three voters vote on the available candidates; in stage four the candidate with the most votes, as measured by the integral over sets of voters that vote for said candidate, is allowed to choose a policy. Ties are resolved against the candidate with financial backing and a citizen that is indifferent between running and not running is assumed not to run.

In this model a citizen is defined entirely by her most preferred policy, which is the bliss point of her utility function. For simplicity we will henceforth use the words "citizen" or "voter" to mean "citizen's MPP" or "voter's MPP". Let there be a continuum of citizens, parameterized by $x \in S \subset \mathbb{R}^N$, S being compact and convex. Let the set of voters be measure one, with density $f(x)$ and let them have spherical preferences: for a policy choice $y \in \mathbb{R}^N$ the utility for citizen x is $U(x, y, I) = u(|x - y|) - Ic, u' < 0$ where I is an indicator for whether the voter runs for election and u is differentiable everywhere. Note that these assumptions correspond to A1-A4, with the addition that voters may run for election at cost c which is additively separable from their utility over the chosen policy.

Each citizen-candidate can choose to run for office and may choose a level of expenditure. Let the set of candidates running be denoted S. All candidates must match the level of expenditure for the highest spending candidate and therefore the actual cost of running is $c = \max_{s \in S} c^*(s)$ where c^* is the cost chosen by candidate s. The citizen chosen by the financial backers can choose to stand for election at no cost because her costs are borne by the backers. She therefore has utility $U = u(|y - x|)$ regardless of whether she runs.

Denote $v = \min_{x,y \in S} u(|x - y|)$. We assume that the value to the financial backers of having their policies implemented, V is greater than v. This will ensure that the financial backers will choose to support some candidate. Assume also that the financial backers' policies are only implemented by a candidate if they supported said candidate. Given that a candidate is a citizen and therefore indifferent to policies of interest to the financial backers, it seems reasonable that she will not help backers who supported her opponent in a preceding election but will help those who helped her.

Citizen-candidate models with a fixed cost of entry have been studied before, and it has been established that citizens will only choose to run if their most preferred policy sufficiently differs from that of alternate candidates. This is because they will only run if the payoff from victory times the probability of victory (which may be 0 or 1) is higher than the cost of running. The payoff from victory is the impact on utility from implementing an MPP over another candidate's MPP. Then if a candidate has staked out a position near a citizen who is deciding whether to enter the race, that citizen will see little benefit to running compared to one whose most preferred policy is farther away.

The set-up presented above has two immediate consequences. First, financial backers will always choose a spending level, c, that is high enough to prohibit entry by potential contenders. Second, there is no way for potential candidates to commit to any policy and therefore all candidates are viewed by voters as standing for their MPPs. In light of these facts we can solve the game backwards fairly easily.

In stage 4 the elected candidate will choose her MPP.

In stage 3 voters vote sincerely for their preferred candidate.[14]

In stage 2, candidates may choose to run against the candidate supported by the financial backers if their expected utility from running is higher than their expected utility from not running. Calculating the set of equilibria here for an arbitrary c is somewhat complex and, in light of the solution in stage 1, unnecessary.

In stage 1 the financial backers will select a candidate and choose some c large enough that no citizen is willing to stand for election in stage 2. Recall that in models with multiple dimensions of policy, once one candidate has staked out a position, there is typically another that can beat that candidate. The financial backers are assumed to value having their policies implemented more than the cost of ensuring implementation and are therefore willing to pay to get their preferred candidate in office.

Their problem in stage 1 is therefore to choose a candidate that minimizes the cost c of preventing electoral competition. A candidate will only stand for election in stage 2 if she could beat the candidate supported in stage 1. Consider the problem of a citizen in stage 2 who could beat the candidate standing for election in stage 1. When she wins the election, she will choose to implement her most preferred policy which is defined to be x. Her utility from running is therefore $u(|x-x|)-c$. If she chooses not to run she receives utility $u(|x-y|)$ where y is the most preferred policy of the candidate chosen in stage 1. Therefore the candidate will run iff $u(|x-x|)-c > u(|x-y|)$. Rearranging,

$$c < u(|x-x|) - u(|x-y|) = u_0 - u(|x-y|)$$

Therefore, in order to ensure that their candidate wins, the financial backers much choose a c where

$$c = \max_{x \in W(y)} u_0 - u(|x-y|)$$

where $W(y)$ is defined as before to be the set of alternative policies that could beat policy y in pair-wise voting. Because the backers wish to minimize

[14] It would be unnatural to argue for any other voting rule than sincere voting without adding something more to this model. After all, given that a voter mas measure 0 and therefore a vote doesn't matter, it is difficult to see why people would bother to vote if not to support their preferred candidate.

this cost while still ensuring victory, their chosen candidate will be x^* satisfying:

$$\begin{aligned} x^* &= \arg\min_{y'} \max_{x \in W(y')} u_0 - u(|x - y'|) \\ &= \arg\min_{y'} \max_{x \in W(y')} |x - y'| \end{aligned}$$

This is the definition of the defensive optimum. Therefore, in a model where financial interests and voters are primarily interested in policies along different dimensions, which is more common than one might initially imagine, the defensive optimum is the policy outcome and the properties of the DO discussed previously apply. That is, we can be sure a solution to the financial backers' problem exists, is unique and continuous. We can also apply our results on minority influence to see how this type of political setting affects the power of various minority groups.

7.2 Incumbency advantage and pork/advertising

In this and the following section I will not provide the same level of detail as in the preceding section regarding disinterested interest groups. Instead, I outline the basic premise and leave it to the reader to more thoroughly develop the model discussed. It is well established that incumbents have a significant advantage in political contests, though there is no consensus on the precise reason for that advantage. Gelman and King (1990) were the first to properly estimate incumbency advantage and establish its existence. Levitt and Wolfram (1997) showed that incumbency advantage is largely due to the ability of incumbents "to deter high quality challengers," a finding supported more recently by Gowrisankaran, Mitchell and Moro (2008). These studies involved elections for the US House of Representatives and US Senate.

While these papers refer to "quality" as though it were something that all voters agree upon, an alternative view is that the quality of a challenger is in the eye of the beholder. The following model offers an interpretation in line with their empirical findings but with a basis acknowledging the heterogeneity in voter preferences. Suppose incumbents can devote time in office delivering federal funds to their constituents or implementing policy. Voters have preferences defined by $U = u(|x - y|) + p$ where x is the voter's bliss point, y is the implemented policy and p is delivered pork. All voters prefer more pork to less because the cost is borne by all states/districts but the benefit accrues only to their state/district. It is costly for the politician to deliver pork so she would prefer to spend her time on policy making, but she is an opportunist and will deliver the pork necessary to win. We also assume here that potential challengers here are opportunistic.

Incumbents are more able to deliver pork because, as discussed previously, positions on important committees like Appropriations and Ways and Means are assigned by seniority. Suppose before each election, the candidate must commit to a policy choice y and level of pork p. As incumbent, the candidate must commit to these levels prior to challengers entering the race. Because delivering pork is costly, she wishes to choose the policy y that minimizes the amount of pork she must deliver to win the election.

Potential competitors cannot deliver as much pork because they lack seniority in the relevant committees, so a competitor staking out a policy "close to" y cannot win an election. The larger the choice of p, the larger this protected radius becomes (and the radius is a sphere because p is additively separable from u, which is spherical). As should be clear, the incumbent's incentives are similar to those of the financial backers in the preceding section and her choice will be to choose y to minimize the pork she must promise to deliver. This choice of y corresponds to the defensive optimum.

It should also be noted that substituting the pork story for an advertising story would not affect the results. Suppose the incumbent must raise money for advertising, which is costly. Voter utility is $U = u(|x - y|) + A$ where A is the level of advertising, and suppose that challengers are citizen-candidates. Then the goal of the incumbent is to choose y to minimize A while still winning the election. Again, the defensive optimum serves this purpose.

In this model there will be no challengers, but the point should be made that this corresponds to the idea of "weak challengers". While in practice challengers do appear, they do not put in a serious effort. Incumbency advantage results from weak challengers because there is no room for strong challengers in either of these incumbency advantage models.

7.3 Entrenched incumbents

Finally, turn to another incumbency advantage story. Suppose voter utility is $U = u(|x - y|) + c$ where c is some constant representing their preference for an incumbent. Each period, a challenger is drawn with alternative policy z from some distribution $f(z)$ to compete against the incumbent.

Because of the shape of preferences, values of z (the challenger's policy) close to the incumbent will not cause the challenger to lose. Not enough voters strongly prefer the challenger to the incumbent to support a challenger victory. Values of z that are too far away, however, would lose to the incumbent even without any incumbency advantage. For c large enough, there will be a set of policies $S(c)$ that cannot lose once ensconced as incumbent. Then if $S(c) \neq \varnothing$ we have $x^* \in S(c)$. Further, this does not hold for any other policy $y \neq x^*$. In words, if there exists a set of policies that cannot be unseated, given c, then the defensive optimum is in that set. The defensive optimum is the only such policy for which this can be said (i.e. there exists a c such that only the defensive optimum cannot be unseated).

8 Conclusion

This paper identified a solution to a variety of models of multidimensional voting called the defensive optimum. This solution was shown to have a variety of attractive characteristics: it always exists, is unique and is continuous in the density of voter preferences. These characteristics allow researchers whose models yield the defensive optimum as the equilibrium policy choice to rest assured that analyses of their models assuming these characteristics are well founded. Furthermore, uniqueness and continuity allow standard methods like regression to be used in empirical work in this space.

The defensive optimum was also shown to have other features that are attractive from a more aesthetic perspective. First, the DO corresponds to the Condorcet winner when one exists. Second, the DO lies on axes about which the density of voter preferences is symmetric. This means that if we add additional dimensions to the policy space where voters' preferences along the new dimension are uncorrelated with their preferences along the old, the DO will remain at the median of the new policy dimension. Third, standard preference distributions (like uniform distributions over a shape like a regular polygon, for example) yield a defensive optimum at the center of these distributions. It seems natural that a chosen policy would be at the center of the voter distribution, especially when the distribution is nicely shaped, but many other solutions to multidimensional voting models do not have this property.

The dependence of the location of the DO on properties of the voter distribution was analyzed in the context of a minority population's influence over policy. Minorities were shown to be more influential when they are larger, more concentrated and more distant from the majority in terms of policy preferences. These facts correspond well to casual observation and also provide some basis for the practice of bloc voting.

Finally, a more detailed analysis of these effects under strong distributional assumptions showed that when the majority population is radially symmetric, very small minorities are three times as powerful, per member, as the larger population. In fact, this shows that the marginal influence of every individual in a population is three times larger than the average influence. It is also shown that the size and concentration of the majority are complementary: the increase in influence as size increases is higher for more concentrated majorities and vice versa. Also, minority size and distance from the majority are complementary.

These results apply to any model yielding the defensive optimum as the resulting policy choice. In the section 7, three such models are presented. First is one in which interest groups and voters are interested in different issues (in elections for state legislature, for example, voters may care most about abortion, the sales taxes, and crime while many donors may be most interested in targeted highway spending, state loan guarantees and other political pork). Embedding this framework in a citizen candidate model yields the defensive optimum as the resulting policy choice. Second is one in which incumbents can devote time in office to "staking out" policy territory near their actual policies and thus excluding competitors from entering a race holding policy positions in this territory. Again, the result in this model is the defensive optimum. Third is one in which incumbents may become entrenched via some fixed incumbency advantage. If incumbency advantage is sufficient for any policy to become entrenched, then the defensive optimum is one such policy. Each model has a flavor of "defensiveness" to it, and indeed many models in which the defensive optimum is selected will involve some candidate, advertiser or donor defending policy turf; hence the term "defensive optimum".

Ideally theorists and empiricists will take advantage of the analysis in this paper to develop models of political institutions or voting rules yielding the defensive optimum as the ultimate policy choice. Existence, uniqueness, continuity and the host of properties described previously make it an attractive outcome and make its analysis straightforward.

9 References

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10 Appendix - Proofs

Proof of Proposition 1. Let $W(x)$ be the set of potential candidates that could beat candidate x in an election. Define $c'(y,x)$ as solving the following equation: $c'(y,x) = u(0) - u(|y-x|)$.

1. Because u is continuous over y and x, so is $c'(y,x)$
2. c' is bounded above because u is continuous and S is compact
3. Because of (1) and (2), $c'(y,x)$ achieves a maximum over any compact set
4. Assumptions A2-A3 imply that $W(x)$ is compact
5. Let $c^*(x) = \max_{y \in W(x)} c'(y,x)$. (3) and (4) imply that $c^*(x)$ exists and is unique. Because the max operator preserves continuity, (1) implies that $c^*(x)$ is continuous. (2) implies that $c^*(x)$ is bounded.
6. The goal is to $\min_{x \in S} c^*(x)$. Since $c^*(x)$ is continuous and bounded over S and S is compact, c^* achieves a minimum.

■

The text identifies the fundamental function, $B(f,x,\theta)$, in definition 2 as the distance from any point x to any f-median $M(x,\theta)$ given some underlying voter density f. We need to understand some properties of this function in order to show uniqueness of the defensive optimum and its continuity.

Lemma 1 *$B(f,x,\theta)$ is differentiable in x*

Proof. Referring to figure 2, the derivative of $B(f,x,\theta)$ is $\sin t$ where t is the angle between the direction of change in x and the f-median $M(f,\theta)$. ■

Lemma 2 *$B(f,x,\theta)$ is continuous in f where the distance between two functions is defined by the supnorm: $\|f'-f\| = \sup_x |f(x) - f'(x)|$.*

Proof. We want to show that as we shift an arbitrary hyper-plane that intersects S, there is some minimum rate of change in mass of f from one side of the hyper-plane to the other. Because S is compact and f is differentiable on S, f attains a minimum $\underline{m} > 0$ and a maximum $\overline{m} < \infty$. Also, since S is bounded, it is entirely contained in a closed ball of some radius L, which is contained in some closed hyper-cube whose sides have length 2L. Choose L to be the minimum possible value for which this is possible (since S is compact there does exist such a minimum). Because S is convex, the integral of f between any two parallel hyper-planes both of which intersect S is strictly positive (f is positive over the area of S between the planes). Let the minimum of that integral over any choice of direction vector θ[15] be $\eta(d)$ where d is the distance between

[15] A choice of θ defines a set of parallel hyper-planes, all of which are orthogonal to θ

the two hyper-planes. It is immediately clear that $\eta(d)$ is *strictly* increasing and continuous in d (and therefore invertible) and ranges from 0 at $d = 0$ to 1 at $d = 2L$. We must show that for any ε there exists a δ such that when $\|f' - f\| < \delta$, $|B(f', x, \theta) - B(f, x, \theta)| \leq \varepsilon$.

The change in mass on either side of a half-plane when $\|f' - f\| < \delta$ must be weakly less than $\frac{1}{2}(2L)^N \delta$ because we use the supnorm metric. The integral of f between any two parallel planes intersecting S at distance d is greater than $\eta(d)$. Define $\delta = \frac{2\eta(\varepsilon)}{(2L)^N}$ and we're done. ■

This result should not be surprising. After all, $B(f, x, \theta)$, given some x, merely describes the location of the f-medians of f. When f changes a small amount, its f-medians are also only able to change a small amount because f is positive over its entire support S, which is convex.

Lemma 3 *$B(f, x, \theta)$ is differentiable in θ*

Proof. Establishing the differentiability of B in θ requires finding a linear operator D such that

$$\lim_{\theta \to \theta_0} \frac{|B(f, x', \theta) - B(f, x, \theta) - DB(f, x, \theta)|}{\|\theta - \theta_0\|} = 0 \tag{13}$$

Recall that $\theta \in [0, \pi)^{N-1}$ is a direction vector: If a point is expressed as $(x_1, x_2, ..., x_N)$ in Cartesian coordinates then it can be expressed as $(\theta_1, \theta_2, ..., \theta_{N-1}, r)$ in radial coordinates. The limit in equation 13 is the limit of a sequences of vectors θ that all lie in a plane: it is a limit from one direction. For example, if we let $\theta \in [0, \pi)^2$ we could approach θ_0 along the second dimension: $\theta_n = (\theta_0^1, \theta_0^2 + c/n)$ and take a limit as n tends to infinity. Figure 6 shows the case where $\theta_0^1 = 0$. Then $\{\theta_n\}$ spans the plane defined by $(x_1, 0, x_3)$ where $x_1, x_3 \in \mathbb{R}$.

Because the cumulative density is continuous as we rotate a plane, small changes in θ yield small changes in the density on each side of the rotated plane. Therefore we will only need to shift the plane a little (the cumulative density is continuous in lateral shifts as well) to restore half the mass of f on each side.

Because f is differentiable it is locally linear. WLOG and for simplicity, label the axes $x_1, x_2, ...$ and orient $\mathbb{R}^N$ so that $\theta_0 = (0, 0, 0, ...)$ (θ_0 is the x_1 axis). Let θ_n be a sequence of vectors $\theta_n = (\alpha_n, 0, 0, ...)$, $\lim_{n\to\infty} \alpha_n = 0$. Then the span of $\{\theta_n\}$ is the plane defined by $(x_1, x_2, 0, 0...)$ with $x_1, x_2 \in \mathbb{R}$. The f-median associated with θ_0, $M(f, \theta_0)$ is composed of points $(B(f, 0, \theta_0), x_2, x_3, ...)$ with $x_i \in \mathbb{R}$ for $i = 2, 3, ..., N$.

We can divide $M(f, \theta_0)$ into halves, one where x_2 is positive and one where it is negative. Let the half of $M(f, \theta_0)$ where x_2 is positive be denoted $M(f, \theta_0)^P$ and the other half be denoted $M(f, \theta_0)^N$. Let the integral over $M(f, \theta_0)$ be denoted $IM(f, \theta_0)$. Finally, let

$$IM(f, \theta_0)^P = C \int_{M(f,\theta_0)^P} f(x)\,|x_2|\,dx \tag{14}$$

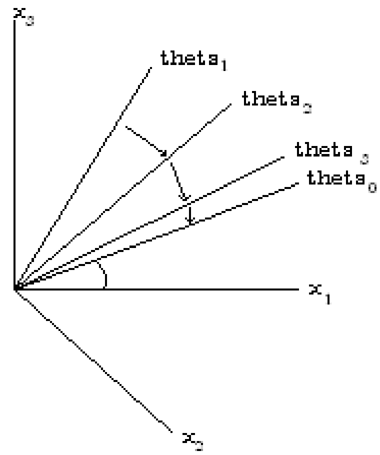


Figure 6: The sequenceof vectors defined by $\theta_n = (0, \theta_0^2 + c/n)$ with limit $\theta_0 = (0, \theta_0^2)$ is contained in the x_1, x_3 plane.

and similarly let

$$IM(f, \theta_0)^N = C \int_{M(f,\theta_0)^N} f(x) \, |x_2| \, dx \tag{15}$$

for some constant C.

As we rotate $M(f, \theta_0)$ a little bit, a triangle (or wedge in more than two dimensions) of voter mass is shifted from the side of M containing x to the side opposite x and a corresponding opposite triangle is moved the other direction. In figure 7, the former is the area labeled A and the latter is the area labeled B. Because f is differentiable, we can approximate the *value* of f at any point in each of these triangles to be the value of f at the closest point on $M(f, \theta_0)$. As is clear from the figure, any given rotation sweeps across a greater area the farther one gets from the point of rotation. In fact, the relationship is linear. We can therefore approximate the net mass that shifts from the same side of M as x to the other side from x as $C \int_{M(f,\theta_0)^P} f(x) \, |x_2| \, dx - C \int_{M(f,\theta_0)^N} f(x) \, |x_2| \, dx = IM(f, \theta_0)^P - IM(f, \theta_0)^N$. ■

This number may be positive or negative. If it is positive then to equate the mass on each side of M we must shift M away from x (to switch some mass of f from the opposite side from x to the same side). If it is negative then we shift M toward x. Either way, the amount of mass that switches sides is equal to the integral over M times the distance of the move: $\left[IM(f, \theta_0)^P + IM(f, \theta_0)^N\right] \times d$.

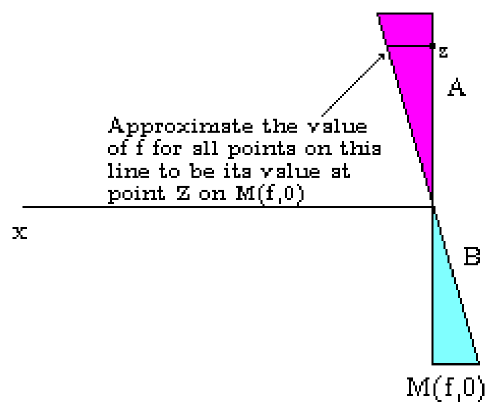


Figure 7: If we rotate plane $M(f,\theta)$ about its intersection with an orthogonal line from x, then the change in mass from the side containing x to the other side is the integral of f over area A minus the integral of f over area B. Because f is differentiable, for small rotations of M, the value of f at a point z in A or B is approximately the value of f at the orthogonal projection of z onto M. Therefore the change in mass is approximately $C\int_{M(f,\theta_0)^P} f(x)\,|x_2|\,dx - C\int_{M(f,\theta_0)^N} f(x)\,|x_2|\,dx$.

M is an f-median when

$$\begin{aligned}\left[IM(f,\theta_0)^P + IM(f,\theta_0)^N\right] \times d &= IM(f,\theta_0)^P - IM(f,\theta_0)^N \\ d &= \frac{IM(f,\theta_0)^P - IM(f,\theta_0)^N}{IM(f,\theta_0)^P + IM(f,\theta_0)^N}\end{aligned}$$

The derivative we are looking for is the value of d so

$$\frac{\partial B(f,x,\theta)}{\partial \theta} = \frac{IM(f,\theta_0)^P - IM(f,\theta_0)^N}{IM(f,\theta_0)^P + IM(f,\theta_0)^N}$$

Definition 11 *A* ***pressure point*** *of x is an element of the* ***pressure set*** *of x (written $PS(f,x)$) where $PS(f,x) = \arg\max_\theta B(f,x,\theta)$. Also, let the set of f-medians orthogonal to the vectors from x to each pressure point in $PS(f,x)$ be denoted $\overline{PS}(f,x)$.*

Let $\omega(f,x) = c^*(x) = \max_{y \in W(x)}[u(0) - u(|y-x|)]$ for a given f.

Lemma 4 *If A is an open half space in $\mathbb{R}^N$ and $x \notin A$ is a defensive optimum, it cannot be the case that $PS(f,x) \subseteq A$*

Proof. If this were the case, then it would be possible to move x a little bit orthogonally toward the boundary of A. This new point, y, is closer to all pressure points of x. $B(f,x,\theta)$ is reduced for some values of θ and increased for others. Let the set of θ for which B is reduced by moving from x to y be called $C(y)$ and the set for which it is increased by called $E(y)$. The defensive optimum x solves

$$\begin{aligned} x &= \arg\min_s \max_\theta B(f,s,\theta) \\ &= \arg\min_s \max\{\max_{\theta\in C} B(f,s,\theta), \max_{\theta \in E} B(f,s,\theta)\}\end{aligned}$$

Because the pressure points of x are in A, $\max_{\theta\in C} B(f,x,\theta) > \max_{\theta\in E} B(f,x,\theta)$. Let the difference between the two be $D(d)$ where $d = |x-y|$ is the distance between x and y. Clearly $\lim_{d\to 0} D(d) > 0$ (since all pressure points are strictly farther from x than any other f-medians) and since $B(f,x,\theta)$ is continuous in x, D is also continuous. Then there exists some small d so that $D(d) > 0$ and $d > 0$. Because at this y, $\max_{\theta\in C} B(f,y,\theta) > \max_{\theta\in E} B(f,y,\theta)$, we have that $\max\{\max_{\theta\in C} B(f,y,\theta), \max_{\theta\in E} B(f,y,\theta)\} = \max_{\theta\in C} B(f,y,\theta)$. By construction, $\max_{\theta\in C} B(f,y,\theta) < \max_{\theta\in C} B(f,x,\theta)$. Therefore x was not a defensive optimum. ■

Understanding the intuition of the above argument is key to understanding the following proofs. For the intuition we can restrict attention to $N = 2$ dimensions of policy. The shaded areas in figure 9 show the set of potential candidates that could beat each of three locations in an election for some voter density f. That is, if x is a candidate who is running for office, any competitor in the shared area associated with x would win an election against x. The

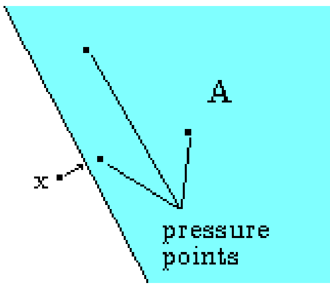


Figure 8: If A is an open half plane not containing x but containing all of x's pressure points, then by moving x a little bit orthogonally toward the boundary of A, the distance to the farthest pressure point declines. x can therefore not be a defensive optimum.

distance from x to the boundary of these areas in a direction θ is $B(f, x, \theta)$. Because B is continuous in x, these sets are "continuous" in x as well, meaning that a small change in x would yield a small change in the shape of these sets (they morph).[16] In picture A of figure 9 x could not be a defensive optimum. If we move a little bit to the right x would get closer to the farthest points on the boundary of this set. As x moves toward a particular location, the boundary at that location also will move toward x (voters between x and the boundary will switch from voting for the boundary candidate to voting for x. As this takes place, the greatest distance between x and the boundary is reduced, and this distance is what we seek to minimize.

At some point as x moves to the right, the set of potential candidates that could beat x looks like picture B in figure 9. In this case, as x moves toward the farthest point in the set that can beat it, it moves away from other candidates that can beat it (and those boundaries also move away from x). Still, a shift up and right, while increasing the distance to some candidates, still reduces the distance to the farthest candidate. In picture C in figure 9, x is a defensive optimum. x is equidistant from the farthest points on each arm of the set that can beat it in an election and there is no way to move toward one of those points without moving farther from another.

Lemma 5 *If A is a closed half space in $\mathbb{R}^N$ and x on the boundary of A is a defensive optimum and $PS(f, x) \subseteq A$ then at least N pressure points must lie on the boundary of A*

Proof. In N dimensions, N points define a N-1 dimensional hyperplane. The boundary of A is a hyperplane (on either side of a hyperplane is a half space). If fewer than N pressure points lie on $bd(A)$ we can rotate $bd(A)$ about x so

[16] We have shown that $B(f, x, \theta)$ is continuous in x. Indeed, as B defines the boundaries of these sets, they will be continuous using many distance metrics.

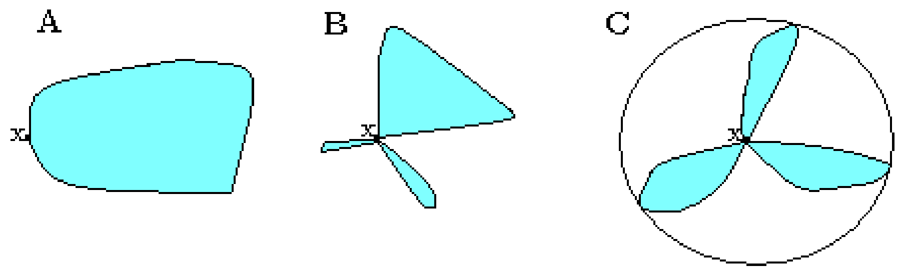


Figure 9: The shaded areas define the set of alternative policies that could beat point x in an election. The distance from x to the boundary of these sets is $B(f, x, \theta)$. In picture A, x could move to the right and reduce the distance to the farthest potential candidate that could beat her. x is therefore not a defensive optimum. In picture C there is no way for x to move that would reduce the distance to the farthest potential candidate who could beat her in an election, so x may be a defensive optimum.

that all pressure points of x are in the interior of the half-plane. Then the open half-space created by removing the boundary contains all pressure points of x but not x itself. By the preceding lemma, x cannot be a defensive optimum. Figure 10 highlights the procedure for $N = 2$. ■

Proof of Proposition 2.

1. Suppose we have a defensive optimum x and $PS(f, x) \subsetneq A$ for any A where A is a closed half space in $\mathbb{R}^N$. This implies that any other policy choice x' is necessarily not an DO. x' is necessarily farther from one of the f-medians contained in $\overline{PS}(f, x)$.

2. Now suppose that $PS(f, x) \subseteq A$ for some closed half-space $A \subset \mathbb{R}^N$. Then a shift to another DO x' must be parallel to the f-medians in the set $\underline{\overline{PS}}(f, x) \equiv \overline{PS}(f, x) \cap Bd(A)$ where $Bd(A)$ is the boundary of A. As in (1), a non-parallel move to x' moves farther away from at least one of those half-planes so x' cannot be a DO.

3. We now see that we can only consider parallel shifts to $\underline{\overline{PS}}(f, x)$. Suppose we have a DO x and a DO candidate x' where $|x - x'| = d$ and the vector between them is parallel to $\underline{\overline{PS}}(f, x)$. We need to show that the distance from x' to at least one half-plane is higher than from x to that same half-plane. Choose an arbitrary pressure point in $\underline{\overline{PS}}(f, x)$ and call it PP. Align the axes so that:

   (a) $x = (0, 0, ..., 0)$ (the origin)

   (b) $x' = (0, d, 0, 0, ..., 0)$ for some number d (i.e. is located on the x_2 axis)

   (c) $PP = (t, 0, 0, ..., 0)$ for some number t (i.e. is on the x_1 axis)

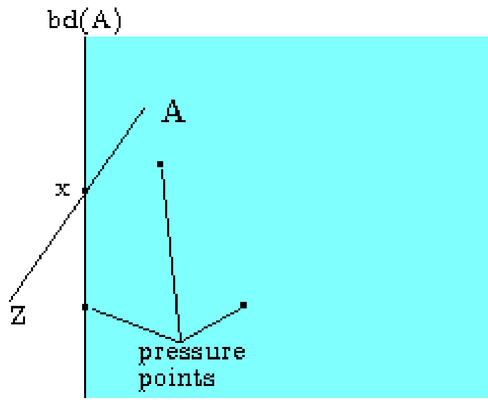


Figure 10: A is a closed half plane containing the pressure points of x which is on its boundary. Only one pressure point lies on the boundary of A. Line Z defines an open half plane that contains all pressure points and not x and by the preceding lemma x cannot be a defensive optimum.

This orientation is always achievable. The shift from x to x' must be orthogonal to the line containing x and PP because of (1) and (2).

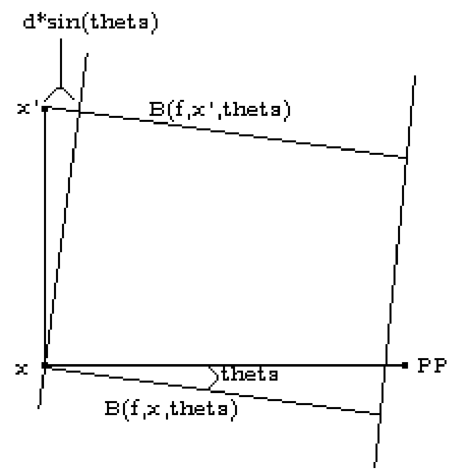
4. Consider a rotation within the x_1, x_2 plane away from x' and we will calculate $B(f, x', \theta)$, as in figure 3. $B(f, x, \theta)$ is maximized at $\theta = 0$ by construction (this it the direction to a pressure point) and, because x is a defensive optimum, any other potential DO cannot have a value for B greater than this. If we can show that $B(f, x', \theta) > B(f, x, 0)$ for some θ, then we have our proof. $B(f, x', \theta) = B(f, x, \theta) + d\sin(\theta)$. Taking a derivative with respect to θ in this direction:

$$\begin{aligned} \frac{\partial B(f, x', \theta)}{\partial \theta} &= \frac{\partial B(f, x, \theta)}{\partial \theta} + \frac{\partial d\sin(\theta)}{\partial \theta} \\ &= \frac{\partial B(f, x, \theta)}{\partial \theta} + d\cos(\theta) \end{aligned} \tag{16}$$

Evaluated at $\theta = 0$ we have

$$\begin{aligned} \frac{\partial B(f, x', \theta)}{\partial \theta} \quad | \quad_{\theta=0} &= \frac{\partial B(f, x, \theta)}{\partial \theta} |_{\theta=0} + d\cos(0) \\ &= 0 + d \\ &= d > 0 \end{aligned} \tag{17}$$

So the distance from x' to some f-median $M(f, \theta)$ close to the f-median defined by $\theta = 0$ must be greater than $B(f, x, 0)$ which means x' cannot be a DO.



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Proof of Proposition 3. $c^*(x)$ is monotonic in the distance to the farthest alternative from x that could beat x in an election. We can, instead of considering properties of c^*, consider properties of $\max_\theta B(f, x, \theta)$.

The function $\omega(f, x) = \max_\theta B(f, x, \theta)$ is continuous in f and x because max preserves continuity. Because the support of x is compact and ω is bounded below by 0, $\min_x \omega(f, x)$ exists. From propositions 1 and 2 we know the DO exists and is unique. As in the body of the paper, let $x^*(f) = \arg\min_x \max_\theta B(f, x, \theta)$ denote the defensive optimum. We aim to show that for every ε there exists a δ such that when $\|f' - f\| < \delta, \|x^*(f') - x^*(f)\| \leq \varepsilon$.

Suppose we create an open ball of radius ε about $x^*(f)$ and call it $C(\varepsilon)$. Then the set $S\diagup C(\varepsilon)$ is compact. Because $\omega(f, x)$ is continuous over this set, it achieves a minimum $m(f, \varepsilon)$. Because $x^*(f)$ is unique, $m(f, \varepsilon)$ is greater than $\omega(f, x^*(f))$. Define

$$\lambda(f, \varepsilon) = m(f, \varepsilon) - \omega(f, x^*(f)) \tag{18}$$

and note that $\lambda(f, \varepsilon)$ is weakly increasing and continuous in ε.

We will briefly move to define another function before returning to $\lambda(f, \varepsilon)$. It was established above that $\omega(f, x)$ is continuous in f. This means that we can define

$$\begin{aligned} G(d) &= \sup_{f, f', x} |\omega(f', x) - \omega(f, x)| \\ s.t. \|f' - f\| &\leq d \end{aligned} \tag{19}$$

and note that $G(d)$ is continuous and $G(0) = 0$. $G(d)$ describes the biggest change in $\omega(f, x)$ for any x, f and f' when f and f' are sufficiently close together. Then $\omega(f', x^*(f)) - \omega(f, x^*(f)) \leq G(d)$. For any point $x \in S\diagup C(\varepsilon), \omega(f', x) - \omega(f, x) \geq -G(d)$.

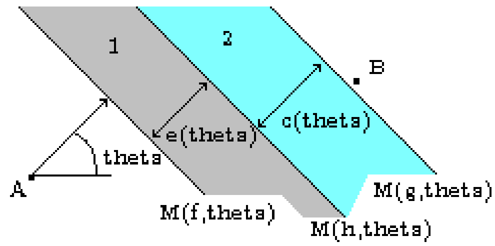


Figure 11: The mass of f in the area labelled 1 is equal to $\Delta^+(f, \theta, e(\theta))$ and the mass of g in the area labelled 2 is equal to $\Delta^+(g, \theta, g(\theta))$. The value of the CDF of h to the left of $M(h, \theta)$ is equal to $(1 - \varepsilon)$ (the mass of f) times the CDF of f at $M(h, \theta)$, which is $1/2 + \Delta^+(f, \theta, e(\theta))$ (mass of f to the left of $M(f, \theta)$ is equal to 1/2 plus ε (the mass of g) times $1 - (1/2 + \Delta^+(g, \theta, c(\theta)))$. $M(h, \theta)$ is only an f-median when the mass of h to the left of M equals 1/2.

Because $G(d)$ is continuous and $G(0) = 0$ and $\lambda(f, \varepsilon)$ is weakly increasing and continuous in ε, for any ε there exists a δ such that $2G(\delta) < \lambda(f, \varepsilon)$ and we have continuity of $x^*(f)$ in f. ■

The intuition of the above proof is straightforward. $B(f, x, \theta)$ is continuous so $\omega(f, x) = \max_\theta B(f, x, \theta)$ is continuous. ω also has a unique minimum over $x \in S$. For any open epsilon ball about $x^*(f)$, we can find a δ small enough that $\omega(f, x^*(f))$ goes up less than $\omega(f, x)$ comes down for any $x \in S \diagup C(\varepsilon)$. Then the minimum of ω must still lie in the epsilon ball.

Proof of Proposition 4. For a given hyperplane $A \in A^*$, $B(f, x, \theta)$ is symmetric across A so if the DO does not lie on A, there is another optimum reflected across A. Because the DO is unique we have a contradiction. Since this is true for any plane $A \in A^*$, it is true for all of them. The defensive optimum lies at their intersection. ■

Proof of Proposition 6. Recall that $\Delta^y(i, \theta, d)$ is defined to be integral between an f-median of g orthogonal to direction vector θ and a parallel hyperplane distance d away in direction y. Here we will define $y = +$ to mean "toward the associated f-median of the other voter density."

The location of an arbitrary f-median in the overall voter density $h_f^g = (1 - \varepsilon)f + \varepsilon g$ is given by

$$\frac{1}{2} = (1 - \varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g, \theta, c(\theta))] \qquad (20)$$

where $e(\theta) + c(\theta) = D(\theta) =$ distance between the f-medians defined by angle θ. Figure 11 highlights why this is true. The mass of f in the area labelled 1 is equal to $\Delta^+(f, \theta, e(\theta))$ and the mass of g in the area labelled 2 is equal to $\Delta^+(g, \theta, g(\theta))$. The value of the CDF of h to the left of $M(h, \theta)$ is equal to $(1-\varepsilon)$

(the mass of f) times the CDF of f at $M(h, \theta)$, which is $1/2 + \Delta^+(f, \theta, e(\theta))$ (mass of f to the left of $M(f, \theta)$ is equal to $1/2$ plus ε (the mass of g) times $1 - (1/2 + \Delta^+(g, \theta, c(\theta)))$. $M(h, \theta)$ is only an f-median when the mass of h to the left of M equals $1/2$.

Equation 20 can be rewritten:

$$\frac{1}{2} = (1-\varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g, \theta, D(\theta) - e(\theta))] \tag{21}$$

This equation implicitly defines the function $e(\theta)$.[17] To follow the next claims, it is important to note that $\Delta^+(f, \theta, s)$ is increasing in s and $\Delta^t(g, \theta, D(\theta) - s)$ is decreasing in s. Fixing e, as ε rises, the RHS of 21 must fall. Therefore, e must compensate by rising.

Now we can write parallel equations for $h_f^g, h_f^{g'}$

$$\begin{aligned} \frac{1}{2} &= (1-\varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e_1(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g, \theta, D(\theta) - e_1(\theta))] & (22) \\ \frac{1}{2} &= (1-\varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e_2(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g', \theta, D(\theta) - e_2(\theta))] & (23) \end{aligned}$$

Because g and g' are equivalently located, $D(\theta)$ is the same in each equation. Equating the two

$$\begin{aligned} & (1-\varepsilon)[\Delta^+(f, \theta, e_1(\theta))] - \varepsilon[\Delta^+(g, \theta, D(\theta) - e_1(\theta))] & (24) \\ = \; & (1-\varepsilon)[\Delta^+(f, \theta, e_2(\theta))] - \varepsilon[\Delta^+(g', \theta, D(\theta) - e_2(\theta))] & (25) \end{aligned}$$

Suppose $e_1 = e_2 = e$. then

$$\Delta^+(g, \theta, D(\theta) - e(\theta)) = \Delta^+(g', \theta, D(\theta) - e(\theta))$$

Because $\Delta^+(g, \theta, s) \geq \Delta^+(g', \theta, s), \forall \theta, s$, this inequality will not hold In fact, the LHS of 24 is smaller that the RHS unless $e_1 \geq e_2$. Our assumption that f and g are separate (implying that f and g' are too) means that the f-medians for the combined distributions must lie between the f-medians for f and g alone. Because we have assumed that g and g' are equivalently located, the f-medians for g and g' are the same. Let an f-median of the distribution h_f^i be denoted $M(hi, \theta)$. $e_1 \geq e_2$ implies $M(hg', \theta)$ lies between $M(hg, \theta)$ and $M(f, \theta)$ for any θ.

The movement property 9 then implies that

$$|x^*(h) - x^*(g)| \leq |x^*(h) - x^*(g')| \tag{26}$$

■

Proof of Corollary 1. Maintaining our definitions and assumptions from proposition.6, let $h_1 = (1-\varepsilon_1)f + \varepsilon_1 g$ and $h_2 = (1-\varepsilon_2)f + \varepsilon_2 g$. That $e(\theta)$ is increasing in ε implies that if $\varepsilon_1 > \varepsilon_2$

$$|M(h_1, \theta) - x^*(g)| \leq |M(h_2, \theta) - x^*(g)| \tag{27}$$

[17]Note that once the densities and size of the population are fixed, the only variable that can be changed is e (which implies c).

9 then implies that

$$|x^*(h_1) - x^*(g)| \leq |x^*(h_2) - x^*(g)| \tag{28}$$

■

Proof of Proposition. 7 Maintaining our definitions and assumptions from proposition 6, let $h = (1-\varepsilon)f + \varepsilon g$ and $h' = (1-\varepsilon)f + \varepsilon g'$. We have

$$\frac{1}{2} = (1-\varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g, \theta, D(\theta) - e(\theta))]$$

and

$$\frac{1}{2} = (1-\varepsilon)[\frac{1}{2} + \Delta^+(f, \theta, e'(\theta))] + \varepsilon[\frac{1}{2} - \Delta^+(g, \theta, D'(\theta) - e(\theta))]$$

where $D(\theta)$ and $D'(\theta)$ refer to distributions associated with g and g' respectively. These two equations define $e(\theta)$ and $e'(\theta)$ implicitly. Because h and h' satisfy 9, we have that $D'(\theta) > D(\theta)$ implying that $e'(\theta) > e(\theta)$. Therefore

$$|M(h, \theta) - x^*(f)| \leq |M(h', \theta) - x^*(f)| \tag{29}$$

and applying 9 again yields

$$|x^*(h) - x^*(f)| \leq |x^*(h') - x^*(f)| \tag{30}$$

■

Proof of Proposition 8. Because f is radially symmetric, $\Delta^y(f, \theta, d) = \Delta(f, d) \forall y, \theta$. That is, beginning at any hyperplane through the center of the radially symmetric density f, the CDF in any direction looks the same. Also let our axes be aligned so that A is the origin and $B = (D, 0, 0, ...)$. That is, the first axis will be labeled to contain both the center of f, A, and the minority atom, located at B. The line connecting these points will be called L and note that the resulting density $h = (1-\varepsilon)f + \varepsilon g$ is laterally symmetric across any hyperplane containing L meaning that the DO will always lie on L. Therefore we can constrain our search for the DO to points on that line.

When $\varepsilon = 0$ the DO is at A because of radial symmetry. For $\varepsilon > 0$ the f-medians fall into three groups. The first group, call it X, is made up of all f-medians that contain L. This group does not move. Less than half the mass now lies on either side of these planes (because a positive atom mass lies on each plane), but the mass on each side is equal regardless of the mass of the atom.

The second group, call it Y, contains all f-medians that do not contain B after moving in response to the introduction of the atom at B. The amount that f-medians in this group shift will solve

$$\begin{aligned} (1-\varepsilon)\Delta(d) &= \frac{\varepsilon}{2} && (31) \\ d &= \Delta^{-1}(\frac{1}{2}\frac{\varepsilon}{1-\varepsilon}) && (32) \\ d &= \Delta^{-1}(\nu) && (33) \end{aligned}$$

where $\nu = \frac{1}{2}\frac{\varepsilon}{1-\varepsilon}$. This is because an atom of size ε will require all f-medians to shift toward the atom so that a volume of voters equal to $\frac{\varepsilon}{2}$ switches sides from the side with the atom to the side with A in order to maintain half the mass on each side.

The third group, call it Z, will be made up of f-medians that shift some, but then hit the atom and get stuck. These f-medians will not have half the mass of h on each side, which is why we did not allow atoms in our earlier work, but will instead have less than half on each side and will contain the atom. Because the DO lies on L and the hyperplanes in this group intersect L at B, more mass will be located on the DO's side of these half planes than on the other side[18] ensuring they could win an election against a candidate reflected across the f-median from them.

The amount that f-medians in group Z shift depends on their angle relative to L. The more they are angled away from L, the more they can shift before hitting the atom. In order to find the location of the DO, we will note that only two sets of f-medians matter in our analysis and that these are the same regardless of the number of dimensions N. These two groups of planes are:

1. $M_0 = M(h, 0)$, the f-median orthogonal to L
2. $M^* = \{M(h, \theta^*) : \theta^* = \cos^{-1}(\frac{1}{D}\Delta^{-1}(\nu))\}$, members of Z that move the same amount as members of Y but contain B after the move.

M_0 is unique and M^* is the set of f-medians in Z where the angle between $M(h, \theta)$ and L is $\cos^{-1}(\frac{1}{D}\Delta^{-1}(\nu))$. All possible candidates of M^* are equally far from any point on L so will all be equidistant from the DO. We will show that the DO y will be equidistant from these two sets of planes:

$$\|y - M^*\| = \|y - M_0\| \tag{34}$$

The point y that equates this distance solves

$$y - d = d\frac{D - y}{D} \tag{35}$$

$$\Rightarrow y = \frac{2dD}{D + d} \tag{36}$$

Then we can plug in for d to get

$$y = \frac{2\Delta^{-1}(\nu)D}{D + \Delta^{-1}(\nu)} \tag{37}$$

In order for this claim to be true, we must prove that all f-medians in X,Y and Z are no farther from y than M_0 and M^* are. Given an arbitrary f-median $\overline{M}$ when $\varepsilon = 0$, the addition of the atom with mass $\varepsilon > 0$ will cause the plane

[18] Since the atom lies on these halfplanes but A does not, there is more mass of f on the side with A (since f is radially symmetric).

to shift so that $\frac{\varepsilon}{2}$ of the mass of f shifts from the side of $\overline{M}$ containing B to the side containing A. This equilibrates the mass of h on each side. The distance d of the shift is therefore given by 31.

This is only true if the atom at B does not get in the way. As was discussed earlier, if the shift of an f-median would require it to cross B, it will get stuck at B. Then the f-median that is able to shift the full amount but just hits B right at the end is a dividing plane. Planes at lower angles to L will hit B before completing the move and get "stuck", and planes with higher angles relative to L will move the full amount. This cut-off plane is found by requiring it to shift the full amount and contain B:

$$\cos(\theta^*) = \frac{d}{D} \tag{38}$$

$$\theta^* = \cos^{-1}(\frac{d}{D}) \tag{39}$$

$$= \cos^{-1}(\frac{1}{D}\Delta^{-1}(\nu)) \tag{40}$$

which is the definition of M^*. For the following exposition, we will refer to angles between half planes and L, not between the orthogonal vector that defines the half plane and L. This is easier to picture and hopefully makes the proof clearer. The two angles are related in that the cosine of the sin of one equals the other.

All members of Z lie closer to any point on L between A and B than M^* because all of the f-medians in Z contain B but have lower angles to L. Basically, these planes are rotated about B toward L relative to planes in M^*. Figure 12 shows this in the case of $\mathbb{R}^2$. The top picture shows the set M^*. In the lower picture, line P1 is at a lower angle to L than the lines in M^*. It is closer to A than the lines in M^* and therefore less mass than $\varepsilon/2$ has moved from the side opposite A to the side containing A. Nonetheless, it has hit atom B and if it moved a little more would contain more than $\varepsilon/2$ extra mass on the side with A. P1 is in set Z. Note that P1 was chosen arbitrarily: All lines through B at angles to L less than those of the lines in M^* will also be in Z.

What about members of Y? Members of Y will intersect L either between A and the DO y or between y and B. The former are defined by angles relative to L between $\sin^{-1}(\frac{d}{y})$ and $\frac{\pi}{2}$ will be at distance $S = y\sin\theta - d$ to y: The intersection of a plane in Y with L will be at distance

$$r = d/\sin\theta$$

from A. The distance of this plane to y is then given by

$$\sin\theta = \frac{S}{y-r}$$

$$= \frac{S}{y - d/\sin\theta}$$

so

$$\sin\theta(y - \frac{d}{\sin\theta}) = S = y\sin\theta - d > 0$$

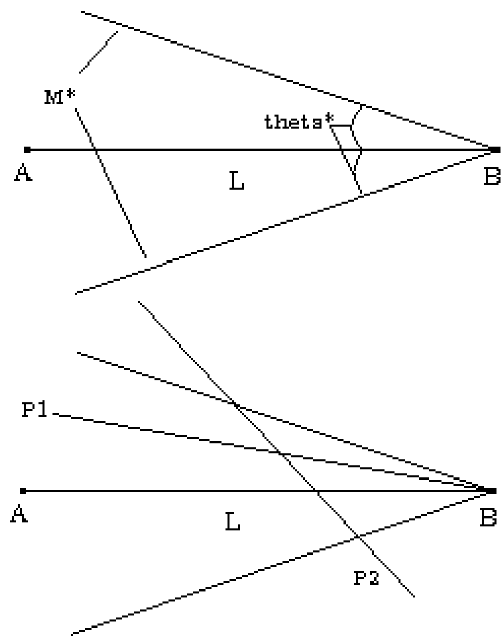


Figure 12: The top picture shows the set M^*. In the lower picture, line P1 is at a lower angle to L than the lines in M^*. It is closer to A than the lines in M^* and therefore less mass than $\varepsilon/2$ has moved from the side opposite A to the side containing A. Nonetheless, it has hit atom B and if it moved a little more would contain more than $\varepsilon/2$ extra mass on the side with A. P1 is in set Z. Line P2 has a sharper angle to A than members of M^* and therefore does not move far enough to intersect B. It is in set Y.

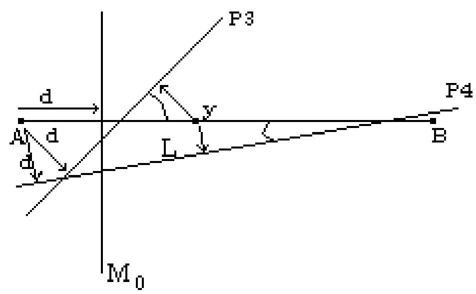


Figure 13: For any f-median in set Y, for example P3, if θ is the angle between the f-median and A, the intersection of the f-median with A is at distance $r = d/\sin\theta$ from A. The distance from y is then $S = |y\sin\theta - d|$

The maximal distance to y attained in this group is for $\theta = \frac{\pi}{2}$ which is associated with f-median M_0. The latter are associated with angles relative to L between $\sin^{-1}(\frac{d}{D})$ and $\sin^{-1}(\frac{d}{y})$. The distance to f-medians in this group is given by $S = d - y\sin\theta$. Within this range of possible angles, the distance is maximized at $\theta = \sin^{-1}(\frac{d}{D})$ which is our definition of M^*.

Figure 13 shows the case for $\mathbb{R}^2$. The intersection of line P3 with L is closer to B than the intersection of M_0 with L. As the angles get flatter (relative to L), that intersection moves to the right. The distance of these f-medians to y is decreasing. At some point an f-median will contain y. For flatter angles the intersection with L proceeds to B and the distance to y is increasing. Once the angle gets flat enough, the f-median will intersect L at B. This is the f-median M^*. As in picture 2 of 12, when angles get even flatter, the f-medians pivot at B and get closer to y again. When the angle between L and the f-median hits 0, the f-median joins group X and has distance to y of 0.

Suppose now that we consider a potential DO y^* on L and that the distance to M_0 is greater than the distance from y^* to M^*. If we move to the left, the former will decrease while the latter will increase. The max of the two will decrease so y^* was not a defensive optimum. The DO must equate the distance to these two f-medians.

The f-median is therefore as defined in equation 37: It is on L at distance $y = \frac{2\Delta^{-1}(\nu)D}{D+\Delta^{-1}(\nu)}$ from A so long as $\frac{2\Delta^{-1}(\nu)D}{D+\Delta^{-1}(\nu)} \leq D \Rightarrow \varepsilon \leq \frac{2\Delta(D)}{1-2\Delta(D)}$. Otherwise the DO is at B. ■

Proof of Proposition 10.

$$\frac{\partial y}{\partial \varepsilon} = \frac{2D^2 \frac{\partial}{\partial \nu}\Delta^{-1}(\nu)}{(D+\Delta^{-1}(\nu))^2}\frac{\partial \nu}{\partial \varepsilon} \quad (41)$$

$$= \frac{D^2}{(D+\Delta^{-1}(\nu))^2(1-\varepsilon)^2} \times \frac{\partial}{\partial \nu}\Delta^{-1}(\nu) \quad (42)$$

For vanishingly small minorities we need to know both $\Delta^{-1}(0)$ and $\frac{\partial}{\partial\nu}\Delta^{-1}(0)$. The integral along any hyperplane through A is constant as a result of radial symmetry. If we rotate that hyperplane in the direction orthogonal to it to an angle of π radians, we will have now integrated over the whole space and therefore this integral must equal $1-\varepsilon$ (the majority population is size $1-\varepsilon$). Therefore the integral over any hyperplane through A equals $(1-\varepsilon)\frac{1}{\pi}$. For very small minorities, since f is differentiable we can assume that it is locally constant: $\Delta(\nu) = (1-\varepsilon)\frac{\nu}{\pi}$. Then

$$\Delta^{-1}(x) = \frac{\pi}{(1-\varepsilon)}x \tag{43}$$

$$\frac{\partial}{\partial x}\Delta^{-1}(x) = \frac{\pi}{(1-\varepsilon)} \tag{44}$$

Then we have

$$\frac{\partial y}{\partial \varepsilon} = \frac{D^2\pi/(1-\varepsilon)}{(D+\frac{\pi}{1-\varepsilon}\frac{1}{2}\frac{\varepsilon}{1-\varepsilon})^2(1-\varepsilon)^2} \tag{45}$$

$$= \frac{D^2\pi}{(D+\frac{\pi}{2}\frac{\varepsilon}{(1-\varepsilon)^2})^2(1-\varepsilon)^3} \tag{46}$$

Evaluated at $\varepsilon = 0$, we get

$$\frac{\partial y}{\partial \varepsilon} = \pi \tag{47}$$

This is proportionate absolute influence. Proportionate relative influence is the quotient of this and distance and equals π/D. ■

As it may interest readers, for radial symmetric densities we can calculate Δ and show that it is concave.

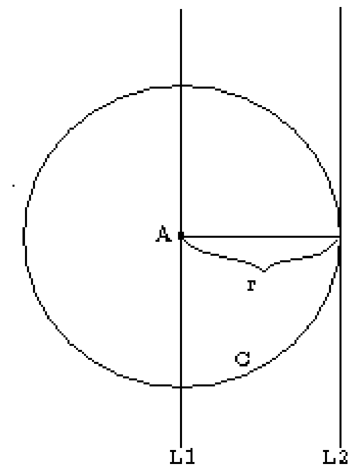
Theorem 1 *If f is radially symmetric, then Δ is concave*

Proof. Consider the case of $\mathbb{R}^2$ shown in figure 10. The integral of f between lines L1 and L2 is equal to $\Delta(r)$. This equal the integral of f along all lines parallel to L1 and L2 and between them:

$$\Delta(r) = \int_0^r \int_{-\infty}^{\infty} f(x_1, x_2)dx_2dx_1$$

$$= 2\int_0^r \int_0^{\infty} f(x_1, x_2)dx_2dx_1$$

Because $f(x)$ is radially symmetric, all points equidistant from A (for example on circle C) have the same value of f. Therefore, the integral along L2



equals the integral along L1 minus the integral along L1 from -r to r. This is true for all lines between L1 and L2, so

$$\Delta(r) = 2\int_0^r \int_{x_1}^{\infty} f(0, x_2) dx_2 dx_1$$

The density associated with the distribution Δ is

$$\begin{aligned} \delta(r) &= \frac{\partial}{\partial r}\Delta(r) \\ &= \frac{\partial}{\partial r} 2\int_0^r \int_{x_1}^{\infty} f(0, x_2) dx_2 dx_1 \\ &= 2\int_r^{\infty} f(0, x_2) dx_2 \end{aligned}$$

Therefore Δ is concave. To get estimates of Δ and two derivatives at $r = 0$ we need one more derivative of δ.

$$\begin{aligned} & \frac{\partial}{\partial r}\left[\delta(0) - 2\int_0^r f(0, x_2) dx_2\right] \\ = & -2\frac{\partial}{\partial r}\int_0^r f(0, x_2) dx_2 \\ = & -2f(0, r) \end{aligned}$$

Then the approximations of $\Delta(0), \Delta'(0)$ and $\Delta''(0)$ are

$$\begin{aligned}
\Delta(0) &= 0 \\
\Delta'(0) &= 2\int_0^\infty f(0, x_2)dx_2 = \frac{1-\varepsilon}{\pi} \\
\Delta''(0) &= -2f(0,0) < 0
\end{aligned}$$

In greater than two dimensions, lines L1 and L2 become hyperplanes, the argument above follows the same logic. Δ is concave. ■